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A study on the mild solution of impulsive fractional evolution equations

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ABSTRACT

This paper is concerned with the formula of mild solutions to impulsive fractional evolution equation. For linear fractional impulsive evolution equations [8–25,27,30,31], described mild solution as integrals over $(t_k, t_{k+1}](k = 1, 2, \ldots, m)$ and $[0, t_1]$. On the other hand, in [26,28,29], their solutions were expressed as integrals over $[0, t]$. However, it is still not clear what are the correct expressions of solutions to the fractional order impulsive evolution equations. In this paper, firstly, we prove that the solutions obtained in [8–25,27,30,31] are not correct; secondly, we present the right form of the solutions to linear fractional impulsive evolution equations with order $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively; finally, we show that the reason that the solutions to an impulsive ordinary evolution equation are not distinct.

1. Introduction

The work of Tai and Wang [1] started a new era of studying fractional-order impulsive evolution equation systems. Then a series of results on fractional-order impulsive differential equations were obtained (see [1–7]). However, the results of authors in [1–7] were proved to be incorrect in [8,32]. Shu et al. [8] and Hernández et al. [32] indicate that the definitions of mild solutions given in [1–7] are obtained by imitating the definitions of mild solutions to first-order nonlinear impulsive evolution equations, and the classical solutions of the impulsive fractional differential equations do not satisfy the definitions of mild solutions presented in [1–7]. Thus, such definitions of the mild solutions are not well-defined. On the other hand, due to the fact that impulsive fractional partial differential equations do not satisfy the semi-group property, it is inappropriate to use the semi-group property to study such equations in [8,32]. Since then, different kinds of nonlinear impulsive fractional evolution equations have been studied with many results obtained in [9–31].

Our work is motivated by recent publications on impulsive fractional evolution equations in [8–31]. For linear fractional impulsive evolution equations with order $0 < \alpha < 1$, [8–25,27,30,31] described its solution as integrals over $(t_k, t_{k+1}](k = 1, 2, \ldots, m)$ and $[0, t_1]$. On the other hand, in [26,28,29], their solutions were expressed as integrals over $[0, t]$. The solutions obtained in [26,28,29] and [8–25,27,30,31] have been further discussed in other literatures. However, it is still not clear what are the correct expressions of solutions to the fractional order impulsive evolution equations. In this paper, we will give an affirmative answer for the above problems, and prove that the solutions obtained in [8–25,27,30,31] are not correct and present the right form of the solutions to such differential equations. Then we further investigate the solution of linear fractional impulsive

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evolution equations with order $1 < \alpha < 2$. Finally, we explain the reason that the solutions to an impulsive evolution equation can not be distinct.

The rest of paper is organized as follows. In Section 2, some notions and notations are presented. In Section 3, we give a theoretical basis by using a series of lemmas. In Section 4, first, we show the formula of solution in [8–25,27,30,31] is incorrect, and prove the incorrectness in theory, and then we show the formula of solution in [26,28,29] is correct, furthermore we investigate the right form of the solutions to linear fractional impulsive evolution equations with order $1 < \alpha < 2$. In Section 5, the reason that the solutions to an impulsive evolution equation are not distinct is given.

2. Preliminaries

In this section, we state some notations and notions which will be used throughout this article.

Definition 2.1 [34]**.** Let $a, \alpha \in \mathbb{R}$. A function f: $[a, \infty) \to X$ is said to be in the space $C_{a,\alpha}$ if there exist a real number $p > \alpha$ and a function $g \in C([a, \infty), X)$ such that $f(t) = t^p g(t)$. Moreover, f is said to be in the space $C^m_{a,\alpha}$ for some positive integer m if $f^{(m)} \in C_{a,\alpha}$.

Definition 2.2 [33]**.** Let $A : \mathscr{D} \subseteq X \to X$ be a closed linear operator. A is said to be sectorial operator of type (M, θ, α, μ) if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the α-resolvent of A exists outside the sector,

$$\mu + S_\theta = \{\mu + \lambda^\alpha : \lambda \in \mathbb{C}, |\mathrm{Arg}(-\lambda^\alpha)| < \theta\}$$

and

$$\|(\lambda^\alpha I - A)^{-1}\| \le \frac{M}{|\lambda^\alpha - \mu|}, \qquad \lambda^\alpha \notin \mu + S_\theta.$$

Definition 2.3. If the function $f \in C^m_{a,\alpha}$ and $m \in \mathbb{N}^+$, the fractional derivative of order $\alpha > 0$ of f in the Caputo sense is given by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \qquad m - 1 < \alpha \le m.$$

In order to research the existence of mild solutions and the qualitative theories of fractional evolution equations, we need help of solution operators $S_\alpha(t)$, $T_\alpha(t)$, $K_\alpha(t)$(see [1 ∼ 34]), where A is a sectorial operator of type (M, θ, α, μ), and

$$S_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda,$$
$$T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda,$$
$$K_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-2} R(\lambda^\alpha, A) d\lambda$$

and c is a suitable path.

Lemma 2.1 [33]**.** *If A is a sectorial operator of type (M, θ, α, μ), then, for $\|S_\alpha(t)\|$, the following estimates hold:*

(i) *If $\mu \ge 0$, then for $\phi \in (\max\{\theta, (1-\alpha)\pi\}, \frac{\pi}{2}(2-\alpha))$, we have*

$$\|S_\alpha(t)\| \le \frac{K_1(\theta,\phi) M e^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}} [(1 + \frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha}} - 1]}{\pi \sin^{1+\frac{1}{\alpha}} \theta} (1 + \mu t^\alpha) + \frac{\Gamma(\alpha) M}{\pi(1+\mu t^\alpha) |\cos\frac{\pi-\phi}{\alpha}|^\alpha \sin\theta \sin\phi},$$

where $t > 0$.

(ii) *If $\mu < 0$, then for $\phi \in (\max\{\frac{\pi}{2}, (1-\alpha)\pi\}, \frac{\pi}{2}(2-\alpha))$, we have*

$$\|S_\alpha(t)\| \le \left(\frac{eM[(1+\sin\phi)^{\frac{1}{\alpha}} - 1]}{\pi |\cos\phi|^{1+\frac{1}{\alpha}}} + \frac{\Gamma(\alpha) M}{\pi |\cos\phi| |\cos\frac{\pi-\phi}{\alpha}|^\alpha} \right) \frac{1}{1 + |\mu| t^\alpha},$$

where $t > 0$.

Lemma 2.2 [33]**.** *If A is a sectorial operator of type (M, θ, α, μ), the following estimates hold:*

(i) *If $\mu \ge 0$, for $\phi \in (\max\{\theta, (1-\alpha)\pi\}, \frac{\pi}{2}(2-\alpha))$, we have*

$$\|T_\alpha(t)\| \le \frac{M[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha}} - 1]}{\pi \sin\theta} (1+\mu t^\alpha)^{\frac{1}{\alpha}} t^{\alpha-1} e^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}} + \frac{M t^{\alpha-1}}{\pi(1+\mu t^\alpha) |\cos\frac{\pi-\phi}{\alpha}|^\alpha \sin\theta \sin\phi},$$