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On the ball of convergence of secant-like methods for non-differentiable operators

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ABSTRACT

In this paper, we analyze the local convergence of a uniparametric family of secant-like methods for solving nonlinear operators in Banach spaces. This new study has an important and novel feature, since it is applicable to non-differential operators. So far, the results of local convergence usually considered can be only applied to differentiable operators.

1. Introduction

We are interested in approximating a solution x^* of a nonlinear equation in Banach spaces,

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \to Y$ and Ω is a non-empty open convex domain in the Banach space X with values in the Banach space Y. In general, the roots of nonlinear Eq. (1) cannot be expressed in a closed form and this problem is commonly carried out applying iterative methods. Thus, starting from one or several initial approximations of a solution x^* of the equation $F(x) = 0$, a sequence $\{x_n\}$ of approximations is constructed such that the sequence $\{\|x_n - x^*\|\}$ is decreasing and a better approximation to the solution x^* is then obtained at every step. Obviously, the interest focuses on $\lim_n x_n = x^*$.

In relation to the above, we can obtain the sequence of approximations $\{x_n\}$ of different ways, depending on the iterative methods that are applied. If the operator F is not differentiable, we have to choose the iterative method carefully. For this case, there are iterative methods, less studied, that do not use derivatives in their algorithms. This type of methods usually use divided differences [10] instead of derivatives. The best known iterative method of this type is the secant method [1,2], whose algorithm is:

$$\begin{cases} x_0, x_{-1} \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{2}$$

where $[u, v; F]$, $u, v \in \Omega$, is a first order divided difference [10], which is a bounded linear operator such that

$$[u, v; F] : \Omega \subset X \longrightarrow Y \quad \text{and} \quad [u, v; F](u - v) = F(u) - F(v).$$

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From an interesting idea appeared in [6] a uniparametric family of secant-like methods for solving (1) is constructed, where the authors propose the following uniparametric family of secant-like methods:

$$\begin{cases} x_0, x_{-1} \text{ given in } D, \quad \lambda \in [0, 1], \\ y_n = \lambda x_n + (1-\lambda)x_{n-1}, \quad n \geq 0, \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1}F(x_n), \end{cases} \tag{3}$$

which are considered as a combination of the secant method and Newton's method, since (3) is reduced to the secant method if $\lambda = 0$ and, provided that F is differentiable, to Newton's method if $\lambda = 1$, since $x_n = y_n$ and $[y_n, x_n; F] = F'(x_n)$. We can also see in [5,7] that the R-order of convergence of (3) is at least $\frac{1+\sqrt{5}}{2}$ if $\lambda \in [0, 1)$, the same order as that of the secant method.

Note that the secant-like methods given by (3) can be considered as a generalization of the secant method. In the real case, we observe that the closer x_n and y_n are, the higher the speed of convergence is. In fact, in [6], the authors show that the higher the value of $\lambda \in [0, 1]$ is, the higher the speed of convergence of (3) is, so that the speed of convergence is close to that of Newton's method when λ is close to 1.

In the study of iterative processes, an important aspect that has been taken into account is the accessibility of the iterative process considered, which shows the domain of starting points from which the iterative process converges to a solution of Eq. (1). The location of starting approximations, from which the iterative methods converge to a solution of the equation, is a difficult problem to solve. This location is from the study of the convergence that is made of the iterative process. In this paper, we focus our attention on the analysis of the local convergence of sequence (3). The local study of the convergence is based on demanding conditions to the solution x^*, from certain conditions on the operator F, and provide the so-called ball of convergence of iterative process, that shows the accessibility to x^* from the initial approximations x_{-1} and x_0 belonging to the ball of convergence.

The extensive literature on the study of the local convergence of derivative-free iterative processes shows a small contradiction. Usually, the known results of local convergence (see [3,4,8,9,11,12] and references therein given) the existence of the operator $[F'(x^*)]^{-1}$, forcing the operator F to be differentiable. These results therefore study the accessibility of the iterative process for differentiable operators. Given that we are considering iterative processes to approximate solutions of non-differentiable operators, it seems logical to study the accessibility in this situation. In this paper, using a new technique of demonstration, we test a result of accessibility to uniparametric family of iterative processes (3) for non-differentiable operators. In addition, as a consequence of our study, we obtain a new result of local convergence for the secant method for non-differentiable operators.

To analyze the local convergence of iterative processes that do not use derivatives in their algorithms, the condition usually required for the operator divided difference is known as Lipschitz continuous condition, which is given by

$$\|[x, u; F] - [y, v; F]\| \leq K(\|x-y\| + \|u-v\|); \quad x, y, u, v \in \Omega. \tag{4}$$

Another condition, under which the local convergence is also usually studied is when the first derivative F' is Hölder continuous in Ω. That is:

$$\|[x, u; F] - [y, v; F]\| \leq K(\|x-y\|^p + \|u-v\|^p); \quad x, y, u, v \in \Omega, \ p \in [0, 1], \tag{5}$$

which generalizes the Lipschitz continuous condition. Note that both conditions involve the operator F to be differentiable [6]. To generalize the above conditions and even to consider situations in which operator F is non-differentiable, we will consider the condition

$$\|[x, u; F] - [y, v; F]\| \leq \omega(\|x-y\|, \|u-v\|); \quad x, y, u, v \in \Omega, \tag{6}$$

where $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous nondecreasing function in its two arguments.

Obviously, we obtain (4) if $\omega(z) = Kz$ and (5) if $\omega(z) = Kz^p$. Moreover, as it is known [6], if $\omega(0, 0) = 0$ then F is a differentiable operator. Therefore, taking into account condition (6), we consider the case in which the operator F is non-differentiable. For example, situations where $\omega(0, 0) \neq 0$, as we can see subsequently.

This paper is organized as follows. In Section 2, we obtain a local convergence result for the secant-like iterative processes given by (3), obtaining their ball of convergence, for non-differentiable operators. In Section 3, from the result of local convergence obtained previously, we analyze the accessibility of secant-like iterative processes from the study of their ball of convergence of a particular case of condition (6) that usually appears in the discretization of differential and integral problems. Finally, in Section 3, we present an application where we illustrate the results obtained for a differential problem of second order which is nonlinear and non-differentiable.

2. A local convergence result for non-differentiable operators

The local convergence results for iterative methods require conditions on the operator F and the solution x^* of Eq. (1). Note that a local result provides what we call ball of convergence and denote by $B(x^*, R)$. From the value R, the ball of convergence gives information about the accessibility of the solution x^*. Now, we analyze the local convergence of method (3).

Firstly, we suppose that there exists a first-order divided difference $[x, y; F] \in \mathcal{L}(X, Y)$, for all pair of distinct points $x, y \in \Omega$, where $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y. Secondly, we suppose:

(A1) Let be x^* a solution of Eq. (1) and consider $\tilde{x} \in \Omega$, with $\|\tilde{x} - x^*\| = \delta > 0$, so that the operator $[x^*, \tilde{x}; F]^{-1}$ exists with $\|[x^*, \tilde{x}; F]^{-1}\| \leq \gamma$.