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# Bivariate Bernstein type operators

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ABSTRACT

In this paper, we introduce bivariate extension of Bernstein type operators defined in [11]. We show that these operators preserve some properties of the original function $f$, such as Lipschitz constant and monotonicity. Furthermore, we present the monotonicity of the sequence of bivariate Bernstein type operators for $n$ when $f$ is $\boldsymbol{\tau}$-convex.



## 1. Introduction

In [11], Cárdenas-Morales et al. presented the following Bernstein type operators for $f \in C[0, 1]$ and $x \in [0, 1]$,

$$\begin{aligned} B_n(f;\tau(x)) &= \sum_{k=0}^{n} \binom{n}{k} (\tau(x))^k (1-\tau(x))^{n-k} (f\circ\tau^{-1})\left(\frac{k}{n}\right) \\ &= B_n^*(f\circ\tau^{-1})\circ\tau \\ &= \sum_{k=0}^{n} \binom{n}{k} (\tau(x))^k (1-\tau(x))^{n-k} f\left(\tau^{-1}\left(\frac{k}{n}\right)\right), \end{aligned} \tag{1.1}$$

where $B_n^*$ is the classical Bernstein operators, $n \in \mathbb{N}$ and $\tau$ is a function defined on $[0, 1]$ and having the properties:
($\tau_1$) $\tau$ is $\infty$-times continuously differentiable on $[0, 1]$
($\tau_2$) $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ on $[0, 1]$.

These conditions ensure that $\tau$ is strictly increasing and the inverse $\tau^{-1}$ of $\tau$ exists on $[0, 1]$. The authors discussed shape preserving and convergence properties and also introduced comparative results. Note that if $\tau(x) = x$, then the operators given by (1.1) reduce to classical Bernstein operators.

Before the construction of bivariate Bernstein type operators, we recall some usual notations and definitions which are essential for our work.

For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0^2$ and $n \in \mathbb{N}$, we will write

$$|\mathbf{x}| := x_1 + x_2, \quad \mathbf{x}^{\mathbf{k}} := x_1^{k_1} x_2^{k_2}, \quad |\mathbf{k}| := k_1 + k_2, \quad \mathbf{k}! := k_1!k_2!$$

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and

$$\binom{n}{\mathbf{k}}=\binom{n}{k_1,k_2}:=\frac{n!}{\mathbf{k}!(n-|\mathbf{k}|)!},\quad \binom{\mathbf{k}}{\mathbf{i}}:=\binom{k_1}{i_1}\binom{k_2}{i_2},\quad \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{k}}:=\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2}.$$

Now let $S:=\{\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2:0\le x_1,x_2\le 1,|\boldsymbol{\tau}(\mathbf{x})|=\tau(x_1)+\tau(x_2)\le 1\}$.

In this paper, for a function $f$ defined on $S$, we consider the bivariate extension of the operators defined by (1.1) as follows:

$$B_n(f;\boldsymbol{\tau}(\mathbf{x}))=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\binom{n}{\mathbf{k}}(\boldsymbol{\tau}(\mathbf{x}))^{\mathbf{k}}(1-|\boldsymbol{\tau}(\mathbf{x})|)^{n-|\mathbf{k}|}(f\circ\boldsymbol{\tau}^{-1})\left(\frac{\mathbf{k}}{n}\right),$$

where $\boldsymbol{\tau}(\mathbf{x}):=(\tau(x_1),\tau(x_2))$ such that $\tau(x_1)$ and $\tau(x_2)$ have the properties given by $(\tau_1)$ and $(\tau_2)$. Moreover, $\boldsymbol{\tau}^{-1}(\mathbf{x}):=(\tau^{-1}(x_1),\tau^{-1}(x_2))$ and $(f\circ\boldsymbol{\tau}^{-1})(\frac{\mathbf{k}}{n})=(f\circ\boldsymbol{\tau}^{-1})(\frac{k_1}{n},\frac{k_2}{n}):=f(\tau^{-1}(\frac{k_1}{n}),\tau^{-1}(\frac{k_2}{n}))$. We observe that these operators are not the tensor product of the operators given by (1.1) obtained by a natural way. If $\boldsymbol{\tau}(\mathbf{x})=\mathbf{x}$, then one obtains the bivariate operators defined on triangle (see [1], p. 109).

**Definition 1.** Let $f$ be a real valued continuous function defined on $D\subset\mathbb{R}^2$ and also let $\tau$ be a function satisfying the conditions $(\tau_1)$ and $(\tau_2)$.

We say that $f$ is a $\boldsymbol{\tau}$-Lipschitz continuous function of order $\mu$ on $D$, if

$$|f(\mathbf{x})-f(\mathbf{y})|\le A\sum_{i=1}^{2}|\tau(x_i)-\tau(y_i)|^{\mu}$$

for $\mathbf{x},\mathbf{y}\in D$ with $A>0$ and $0<\mu\le 1$. We denote the set of $\boldsymbol{\tau}$-Lipschitz continuous functions by $Lip_A^{\boldsymbol{\tau}}(\mu,D)$.

When $\tau(x)=x$, $Lip_A^{\boldsymbol{\tau}}(\mu,D)$ reduces to the Lipschitz class defined in [10].

**Definition 2.** A real valued continuous function $f(\mathbf{x})$ is said to be convex on the convex set $D\subset\mathbb{R}^2$, if

$$f\left(\sum_{i=1}^{r}\alpha_i\mathbf{x_i}\right)\le\sum_{i=1}^{r}\alpha_i f(\mathbf{x_i})$$

for any $\mathbf{x_1},\mathbf{x_2},\dots,\mathbf{x_r}$ in $D$ and for any non-negative numbers $\alpha_1,\alpha_2,\dots,\alpha_r$ such that $\alpha_1+\alpha_2+\dots+\alpha_r=1$ (see [13]).

**Definition 3.** We call $f$ is a convex with respect to $\boldsymbol{\tau}$ or $\boldsymbol{\tau}$-convex in $D\subset\mathbb{R}^2$, if $f\circ\boldsymbol{\tau}^{-1}$ is convex in the sense of Definition 2.

Note that this definition is an analogue of the $\tau$-convexity given in [11].

## 2. Main results

In [10], Cao et al. introduced multivariate Baskakov operators and showed that these operators preserve some properties of the original function $f$ such as monotonicity, semi-additivity and Lipschitz constant. Furthermore, they studied monotonicity of the sequence of multivariate Baskakov operators for $n$ when the attached function $f$ is convex. Finally, the authors computed the rate of approximation with the help of the $K$-functional and modulus of smoothness.

In this part, inspired by the work [10], we firstly show that the Lipschitz constant and monotonicity properties of function $f$ can be retained by the operators $B_n(f;\boldsymbol{\tau}(\mathbf{x})):=B_n^{\boldsymbol{\tau}}f$. After that, we study monotonicity of the sequence of bivariate operators $B_n^{\boldsymbol{\tau}}f$ for $n$ when $f\circ\boldsymbol{\tau}^{-1}$ is convex. For some preservation and approximation properties of univariate or multivariate Bernstein and some other classical positive linear operators we refer the papers [2–10,12–21] and references therein.

**Theorem 4.** *If $f\in Lip_A^{\boldsymbol{\tau}}(\mu,S)$, then $B_n^{\boldsymbol{\tau}}f\in Lip_A^{\boldsymbol{\tau}}(\mu,S)$ for all $n\in\mathbb{N}$.*

**Proof.** Let $\mathbf{x},\mathbf{y}\in S$ and $\mathbf{x}\le\mathbf{y}$ which means that $x_1\le y_1$ and $x_2\le y_2$. Using the definition of the operators $B_n^{\boldsymbol{\tau}}f$ and Binomial formula, we get

$$\begin{aligned}B_n(f;\boldsymbol{\tau}(\mathbf{y}))&=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\binom{n}{\mathbf{k}}(\boldsymbol{\tau}(\mathbf{y}))^{\mathbf{k}}(1-|\boldsymbol{\tau}(\mathbf{y})|)^{n-|\mathbf{k}|}(f\circ\boldsymbol{\tau}^{-1})\left(\frac{\mathbf{k}}{n}\right)\\&=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\binom{n}{\mathbf{k}}((\boldsymbol{\tau}(\mathbf{y})-\boldsymbol{\tau}(\mathbf{x}))+\boldsymbol{\tau}(\mathbf{x}))^{\mathbf{k}}(1-|\boldsymbol{\tau}(\mathbf{y})|)^{n-|\mathbf{k}|}(f\circ\boldsymbol{\tau}^{-1})\left(\frac{\mathbf{k}}{n}\right)\\&=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{k}}\binom{n}{\mathbf{k}}\binom{\mathbf{k}}{\mathbf{i}}(\boldsymbol{\tau}(\mathbf{x}))^{\mathbf{i}}(\boldsymbol{\tau}(\mathbf{y})-\boldsymbol{\tau}(\mathbf{x}))^{\mathbf{k}-\mathbf{i}}(1-|\boldsymbol{\tau}(\mathbf{y})|)^{n-|\mathbf{k}|}(f\circ\boldsymbol{\tau}^{-1})\left(\frac{\mathbf{k}}{n}\right).\end{aligned}$$

If we change the order of the above summations, then we can write

$$B_n(f;\boldsymbol{\tau}(\mathbf{y}))=\sum_{i_1=0}^{n}\sum_{k_1=i_1}^{n}\sum_{i_2=0}^{n-k_1}\sum_{k_2=i_2}^{n-k_1}\binom{n}{\mathbf{k}}\binom{\mathbf{k}}{\mathbf{i}}(\boldsymbol{\tau}(\mathbf{x}))^{\mathbf{i}}(\boldsymbol{\tau}(\mathbf{y})-\boldsymbol{\tau}(\mathbf{x}))^{\mathbf{k}-\mathbf{i}}(1-|\boldsymbol{\tau}(\mathbf{y})|)^{n-|\mathbf{k}|}(f\circ\boldsymbol{\tau}^{-1})\left(\frac{\mathbf{k}}{n}\right).$$