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Computing {2,4} and {2,3}-inverses by using the Sherman–Morrison formula

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ABSTRACT

A finite recursive procedure for computing {2,4} generalized inverses and the analogous recursive procedure for computing {2,3} generalized inverses of a given complex matrix are presented. The starting points of both introduced methods are general representations of these classes of generalized inverses. These representations are formed using certain matrix products which include the Moore–Penrose inverse or the usual inverse of a symmetric matrix product and the Sherman–Morrison formula for the inverse of a symmetric rank-one matrix modification. The computational complexity of the methods is analyzed. Defined algorithms are tested on randomly generated matrices as well as on test matrices from the Matrix Computation Toolbox.

1. Introduction

A rank-one modification (or perturbation) $M = A + bc^*$ of a matrix $A \in \mathbb{C}^{m\times n}$ is based on the usage of two vectors $b \in \mathbb{C}^{m\times 1}$ and $c \in \mathbb{C}^{n\times 1}$. The Sherman–Morrison formula gives a relationship between the inverses M^{-1} and A^{-1} (see, for example, [12]):

$$M^{-1} = A^{-1} - \left(1 + c^*A^{-1}b\right)^{-1} A^{-1}bc^*A^{-1}. \tag{1.1}$$

The identity (1.1) provides a computationally cheap rule for computing M^{-1} in the case when A^{-1} is known in advance. It is well known that the rank-one nature of the Sherman–Morrison formula allows us to compute the inverse of an $n \times n$ matrix in n^2 operations while we need n^3 operations to compute the inverse of a matrix from scratch. The Sherman–Morrison formula (1.1) has been used in finding solutions of a non-linear algebraic equation (see [3]) as well as in the matrix inversion (see [17]). Further, (1.1) has been used in the definition of symmetric rank-one update quasi-Newton method for solving nonlinear unconstrained minimization problems or in finding a root of a function. In the case when at least one of the matrices M or A is singular, it is

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necessary to use the pseudoinverses $A\dagger$ and $M\dagger$. Corresponding generalization of the Sherman–Morrison formula is proposed by Meyer in [19] (see also [4]). A particular case when a nonsingular matrix A is modified to a singular M is considered in [28]. Also, applications of this problem in mathematical statistics are pointed out in [28]. An application of (1.1) in the theory of experimental designs was discussed in [14, Section 8.3]. The various relationships between inner generalized inverses $A^{(1)}$ and $M^{(1)}$ was investigated in [1].

A generalization of (1.1) in the form

$$M^{-1} = (A - UV)^{-1} = A^{-1} + A^{-1}U(I - VA^{-1}U)^{-1}VA^{-1},$$

where$U$ and V dimensionally compatible matrices such that $I - VA^{-1}U$ is invertible, is known as the Sherman–Morrison–Woodbury formula. Its history and its various applications to statistics, networks, structural analysis, asymptotic analysis, optimization, and partial differential equations are discussed in [15]. An algorithm based on the usage of the capacitance-matrix method and the Sherman–Morrison–Woodbury formula was considered in [18]. The algorithm is designed for solving the surface smoothing problem The Sherman–Morrison formula has been often employed in quasi-Newton methods for finding a root of a function or for performing an unconstrained minimization. The case when the pseudoinverses $A\dagger$ and $M\dagger$ are used was solved by Meyer [19] (see also [4]). Trenkler in [28] considered a particular case when a nonsingular matrix A is modified to a singular M and pointed out some connections of this problem with mathematical statistics. Application of (1.1) in the theory of experimental designs was discussed in [14, Section 8.3]. The various relationships between generalized inverses $A^{(1)}$ and $M^{(1)}$ was investigated in [1]. The Sherman–Morrison–Woodbury formula is generalized to the {2}-inverse case in [11].

The computation of the generalized inverse of the rank-one modified matrix has been investigated in [4–7]. The authors of the paper [5] introduced a computational method for the Moore–Penrose inverse of a symmetric rank-one perturbed matrix. As a continuation of such research, a finite method for computing the minimum-norm least-squares solution of the linear system $Ax = b$ was proposed in [6].

In the present paper, we continue the investigations from [27]. Two dual algorithms, designed for computing {2,4}-inverses and {2,3}-inverses, are derived following this approach. The methods derived in [27] initiated two computational methods and algorithms for computing matrix products involving the Moore–Penrose inverse of certain symmetric matrices.

Alternative representations of these generalized inverses are used in this article. Derive algorithms require both the usual inverse and the Moore–Penrose inverse of symmetric rank-one modified matrices.

The rest of the paper is organized as follows. Some basic notations are restated in Section 2. The motivation of the paper is also presented in the same section. Finite recursive algorithms for computing {2,4} and {2,3}-inverses, based on the Sherman–Morrison formula and the Moore–Penrose inverse of symmetric rank-one updates, are presented in Section 3. The computational complexity of defined algorithms is investigated in Section 4. Numerical results are presented and analyzed in Section 5. Rounding error analysis is analyzed in the last section.

2. Preliminaries and motivation

General representations of {2,4} and {2,3}-inverses of prescribed rank, as developed in [8,10,21–23], are restated in Proposition 2.1.

Proposition 2.1. *Let $A \in \mathbb{C}_r^{m\times n}$ be an arbitrary matrix and $0 < s \le r$ be a selected integer. The following general representations of {2,4} and {2,3}-inverses with prescribed rank hold:*

$$\text{(a)}\quad A\{2,4\}_s = \{(VA)^\dagger V \mid V \in \mathbb{C}^{s\times m}\}; \tag{2.1}$$

$$\text{(b)}\quad A\{2,3\}_s = \{U(AU)^\dagger \mid U \in \mathbb{C}^{n\times s}\}. \tag{2.2}$$

The sets $A\{2,4\}_s$ and $A\{2,3\}_s$ are exactly distinguished as outer inverses with prescribed range and null space in the following Proposition 2.2, restated from [26].

Proposition 2.2 [26]**.** The following representations hold, for an arbitrary matrix $A \in \mathbb{C}_r^{m\times n}$ and arbitrary integer s satisfying $0 < s \le r$

$$\text{(a) } A\{2,4\}_s = \left\{A^{(2,4)}_{\mathcal{N}(VA)^\perp,\mathcal{N}(V)} = (VA)^*(VA(VA)^*)^{-1}V \mid V \in \mathbb{C}_s^{s\times m},\ \operatorname{rank}(VA) = \operatorname{rank}(V)\right\}; \tag{2.3}$$

$$\text{(b) } A\{2,3\}_s = \left\{A^{(2,3)}_{\mathcal{R}(U),\mathcal{R}(AU)^\perp} = U((AU)^*AU)^{-1}(AU)^* \mid U \in \mathbb{C}_s^{n\times s},\ \operatorname{rank}(AU) = \operatorname{rank}(U)\right\}. \tag{2.4}$$

Various representations of {2,3} and {2,4}-inverses with prescribed range and null space are considered in [13,29,30]. The effective numerical methods which are based on several modifications of the hyper-power method were introduced in [22]. An adaptation of the successive matrix squaring algorithm from [24] is developed in [26].

Our basic motivation is based on the results derived in [27]. Namely, the pseudoinverse representation (13) from [25]:

$$(VA)^\dagger = ((VA)^*(VA))^\dagger(VA)^*, \quad V \in \mathbb{C}^{s\times m}$$

was used in [27] in conjunction with the representation (2.1) of {2,4}-inverses. In the present paper, we are going to use the alternative representation of {2,4} inverses, stated in (2.3).