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On an integral-type operator from the Bloch space to mixed norm spaces

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ABSTRACT

Let φ be a holomorphic self-map of B and $g \in H(B)$ such that $g(0) = 0$, where $H(B)$ is the space of all holomorphic functions on the unit ball B of $\mathbb{C}^n$. In this paper we investigate the following integral-type operator

$$\mathcal{D}_\varphi^g f(z) = \int_0^1 \mathcal{D}f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(B),$$

where $\mathcal{D}f$ is the fractional derivative of $f \in H(B)$. The boundedness and compactness of the operators $\mathcal{D}_\varphi^g$ between mixed norm spaces and Bloch spaces in the unit ball are studied.

1. Introduction

Let B denote the unit ball and S the unit sphere in $\mathbb{C}^n$. We denote by $H(B)$ the space of all holomorphic functions on B. Let $\Re f$ stands for the radial derivative of f, that is,

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z), \quad z = (z_1, z_2, \ldots, z_n) \in B.$$

Let $f \in H(B)$ with homogeneous expansion $f = \sum_{k=0}^\infty f_k$. The fractional derivative $\mathcal{D}f$ is defined as follows:

$$\mathcal{D}f(z) = \sum_{k=0}^\infty (k+1) f_k(z).$$

Note that $\Re f = \sum_{k=0}^\infty k f_k$, hence $\mathcal{D}f = \Re f + f$.

An $f \in H(B)$ is said to belong to Bloch space, if (see [48])

$$b(f) = \sup_{z \in B} (1 - |z|^2)\left|\Re f(z)\right| < \infty.$$

We denote the Bloch space by $\mathcal{B} = \mathcal{B}(B)$. It is well-known that $\mathcal{B}$ is a Banach space with the norm $\|f\|_{\mathcal{B}} = |f(0)| + b(f)$. From [3] we known that $f \in \mathcal{B}$ if and only if

$$d(f) = \sup_{z \in B} (1 - |z|^2)\left|\mathcal{D}f(z)\right| < \infty.$$

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Moreover,

$$\|f\|_{\mathcal{B}} \asymp \sup_{z\in B}(1-|z|^2)|\mathcal{D}f(z)|. \tag{1}$$

Let ν be a normal function on $[0, 1)$ (see [20]). For $0 < p, q < \infty$, the mixed norm space $H(p, q, \nu) = H(p, q, \nu)(B)$ consists of all $f \in H(B)$ such that

$$\|f\|_{H(p,q,\nu)} = \left(\int_0^1 M_q^p(f,r)\frac{\nu^p(r)}{1-r}dr\right)^{1/p} < \infty,$$

where

$$M_q^q(f,r) = \int_S |f(r\zeta)|^q d\sigma(\zeta),$$

and $d\sigma$ is the normalized surface measure on the unit sphere S.

Let φ be a holomorphic self-map of the unit ball. The composition operator, denoted by C_φ, is defined by $(C_\varphi f)(z) = (f\circ\varphi)(z)$, where $f \in H(B)$. See [4] for some older results on the composition operator C_φ on various function spaces in the unit disk or the unit ball. Approximately after the publication of paper [5] started interest in studying various product-type operators. At first, products of composition and differentiation operators were studied on some spaces of holomorphic functions on the unit disk (see, for example, [9,15,19,27,31,34,41]). For some generalizations of products of composition and differentiation operators on the unit disk, see, for example, [33,40,44,45,49]. Products of composition and integral-type operators on the unit disk were introduced and considerably studied by S. Li and S. Stević somewhat later, see, for example, [11,12,17,22,32]. The last products were generalized independently by S. Stević and X. Zhu, for the case of the unit ball, by introducing the operator

$$C_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \tag{2}$$

where φ is a holomorphic self-map of B and $g \in H(B)$, (see, for example, [16,26,29,36,37,42,46,49,50]), which is an extension of operator L_g defined by

$$L_g f(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}, \quad f \in H(B). \tag{3}$$

The operator was studied, for example, in [10,13,14].

Let φ be a holomorphic self-map of B and $g \in H(B)$ such that $g(0) = 0$. In [8], S. Li introduced the following integral-type operator,

$$\mathcal{D}_\varphi^g f(z) = \int_0^1 \mathcal{D}f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(B), \quad z \in B. \tag{4}$$

For a natural counterpart of operator L_g and its products with composition operators on the unit ball see, for example, [1,2,6,7,23–25,28,30,35,38,39,42,43,47].

Motivated by above mentioned papers here we completely characterize the boundedness and compactness of the integral-type operator $\mathcal{D}_\varphi^g$ between the mixed norm space and the Bloch space in the unit ball.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A\approx B$ means that there is a positive constant C such that $C^{-1}B \le A \le CB$.

2. Main results and proofs

In this section we will give our main results and proofs. First we state several auxiliary results which we will use in the proofs of the main results.

Lemma 1. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then*

$$\Re[\mathcal{D}_\varphi^g(f)](z) = \mathcal{D}f(\varphi(z))g(z), \quad f \in H(B).$$

The proof of Lemma 1 follows from an argument given in [6] and was explicitly stated and proved in this form in [8].

Lemma 2. *([3]) There exists a positive integer $M = M(n)$ with the following property: there exist functions $f_i \in \mathcal{B}$ $(1 \le i \le M)$ such that*

$$\sum_{i=1}^M |\mathcal{D}f_i(z)| \ge \frac{1}{1-|z|}, \quad z \in B.$$

The following criterion for compactness follows from standard arguments similar to those outlined in Proposition 3.11 of [4].

Lemma 3. *Let $X = H(p, q, \nu)$ or $\mathcal{B}$ and $Y = H(p, q, \nu)$ or $\mathcal{B}$, φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then $\mathcal{D}_\varphi^g : X \to Y$ is compact if and only if $\mathcal{D}_\varphi^g : X \to Y$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in X which converges to zero uniformly on compact subset of B as $k \to \infty$, we have $\|\mathcal{D}_\varphi^g f_k\|_Y \to 0$ as $k \to \infty$.*