Applied Mathematics and Computation 273 (2016) 656–663

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Exponential growth with L^p-norm of solutions for nonlinear heat equations with viscoelastic term

Le Xuan Truong[a,*], Nguyen Van Y[b]

[a] Department of Mathematics and Statistics, University of Economics Ho Chi Minh City, 59C Nguyen Dinh Chieu Street, District 3, Ho Chi Minh City, Vietnam
[b] Department of Fundamental Sciences, HCMC University of Food Industry, 140 Le Trong Tan Street, Tan Phu District, Ho Chi Minh City, Vietnam

ARTICLE INFO

MSC :
35K05
35K65

Keywords:
Exponential growth
Viscoelastic
Relaxation function

ABSTRACT

In this work, we study an initial boundary value problem related to the nonlinear viscoelastic equation

$$\big(1 + a|u|^{q-2}\big)u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = b|u|^{p-2}u.$$

We show the exponential growth of solutions with L^p-norm for negative or positive initial energy.

1. Introduction

In this paper, we study the following initial-boundary value problem of a class of viscoelastic equations with multiple nonlinearities

$$\begin{cases} \big(1 + a|u|^{q-2}\big)u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = b|u|^{p-2}u, & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u = u_0 \quad \text{on} \quad \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset in $\mathbb{R}^n$ with smooth boundary; a, b are given positive constants and $p, q > 2$. The relaxation function g satisfies some conditions that will be specified later.

In the physical point of view, this type of problems arises usually in viscoelasticity. In particular, in the absence of nonlinear diffusion term $|u|^{q-2}u_t$, the first equation in (1.1) is reduced to the following equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = b|u|^{p-2}u. \tag{1.2}$$

This equation is the mathematical model of many natural phenomena's in physic science and engineering. For example, in the study of heat conduction in materials with memory, from the heat balance equation the temperature $u(x, t)$ will satisfy Eq. (1.2).

Problems related to Eq. (1.2) have attracted a great deal of attention in last several decades. There have been many results on the existence, blow-up or asymptotic behavior of solutions. For instance in [9], Messaoudi studied the Eq. (1.2) associated with

* Corresponding author. Tel.: +84 98512332.
E-mail addresses: lxuantruong@gmail.com (L.X. Truong), nguyenvanyhv@gmail.com (N. Van Y).

http://dx.doi.org/10.1016/j.amc.2015.10.017
0096-3003/

homogeneous Dirichlet boundary condition. If the relaxation function g is assumed to be nonnegative; $g'(t) \leq 0$ and

$$\int_0^{\infty} g(s)ds < \frac{p-2}{p-3/2},$$

he proved the blow-up of weak solution with positive initial energy by the convexity method. We refer to [3,5,10] for further results on this type of equations.

On the other hand, the first equation in (1.1) without viscoelastic term (that is, the relaxation function g vanishes) can be seen as a special case of doubly nonlinear parabolic-type equations, or the porous medium equations

$$(\varphi(u))_t - \Delta u = |u|^{p-1}u, \tag{1.3}$$

if we take $\varphi(u) = u + |u|^{q-2}u$. We note that for $\varphi(u) = u^m$, by letting $v = \varphi(u) = u^m$, the Eq. (1.3) also can be put into the following equation

$$v_t - \Delta\psi(v) = f(v). \tag{1.4}$$

The blow-up analysis for solutions of quasilinear parabolic systems associated to various boundary conditions has been studied by many authors (see [1,2,8] and references therein).

From mathematical point of view, problem (1.1) is a generalization of (1.2) and (1.3). The presence of the nonlinear diffusion term $|u|^{q-2}u_t$ caused some difficulties in obtaining the priori estimates. Moreover, we note that the problem (1.1) cannot be put into the form (1.4) by changing of variable. In [11], Polat proved a blow up result for the solution with vanishing initial energy of problem (1.1) when $g = 0$ and $n = 1$. In the case of the viscoelastic term is replaced by the term Δu_t, the authors (see [13]) showed the exponential growth of solutions for problem (1.1) with negative or positive initial energy by constructing differential inequalities. This case has been also considered in [6,7].

Our main goal in this paper is to extend the above results to problem (1.1). We will investigate the exponential growth of solutions of (1.1) with negative or positive initial energy. In fact, it will be proved that the L^p-norm of the solution grows as an exponential function. Here we use the idea used in [4,5] which is based on a small perturbation of the total energy. We believe that the current exponential growth results are a step in the direction of blow-up in finite time.

2. Notations, assumptions and preliminaries

In this paper, by $\|\cdot\|_p$ we denote the norm in $L^p(\Omega)$. We will also use the embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

for $p \in [2, p^*]$, where

$$p^* = \begin{cases} \frac{2n}{n-2} & \text{if} \quad n \geq 3, \\ +\infty & \text{if} \quad n = 1, 2. \end{cases}$$

In order to obtain the results, we introduce the following assumptions:

(H1) The constants p and q satisfy the conditions

$$2 < q < p \leq \frac{2n}{n-2} \quad \text{if} \quad n \geq 3; \quad 2 < q < p < +\infty \quad \text{if} \quad n = 1, 2.$$

(H2) The relaxation function $g : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to $C^1(\mathbb{R}^+)$ and satisfies the conditions

(i) $g(0) \geq 0, \quad l := 1 - \int_0^{\infty} g(s)ds > 0, \quad g'(t) \leq 0,$

(ii) $\int_0^{\infty} g(s)ds < \frac{p-2}{p-2+p^{-1}}.$

Remark 2.1. The condition (H1) is needed so that $|u|^{p-2}u \in L^2(\Omega)$. Condition $1 - \int_0^{\infty} g(s)ds = l > 0$ is necessary to guarantee the parabolicity of the system (1.1).

Remark 2.2. Since the function $p \mapsto \dfrac{p-2}{p-2+p^{-1}}$ is continuous and increasing on $(2, +\infty)$, it follows from (H2)-(ii) that

$$\int_0^{\infty} g(s)ds < \frac{\widehat{p}-2}{\widehat{p}-2+\widehat{p}^{-1}},$$

for some real number $\widehat{p}$ in the interval $(2, p)$. Moreover, in compare with [9], the condition (H2)-(ii) is actually an improvement related to $\int_0^{\infty} g(x)dx$.

Similarly to [13], we give a definition for a solution of (1.1).

Definition 2.3. The function $u(x, t)$ is said to be a solution of problem (1.1) on $\Omega \times [0, T)$ if

$$u \in C\big(0, T; H_0^1(\Omega)\big), \quad u_t \in C\big(0, T; L^2(\Omega)\big), \quad |u|^{q-2}u_t \in L^2(\Omega \times [0, T)),$$