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A new nonconforming mixed finite element scheme for second order eigenvalue problem

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A R T I C L E   I N F O

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A B S T R A C T

A new nonconforming mixed finite element method (MFEM for short) is established for the Laplace eigenvalue problem. Firstly, the optimal order error estimates for both the eigenvalue and eigenpair (the original variable u and the auxiliary variable $\vec{p} = \nabla u$) are deduced, the lower bound of eigenvalue is estimated simultaneously. Then, by use of the special property of the nonconforming EQ_1^{rot} element (the consistency error is of order $O(h^2)$ in broken H^1-norm, which is one order higher than its interpolation error), the techniques of integral identity and interpolation postprocessing, we derive the superclose and superconvergence results of order $O(h^2)$ for u in broken H^1-norm and $\vec{p}$ in L^2-norm. Furthermore, with the help of asymptotic expansions, the extrapolation solution of order $O(h^3)$ for eigenvalue is obtained. Finally, some numerical results are presented to validate our theoretical analysis.

1. Introduction

In this paper, we discuss the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain with Lipschitz continuous boundary $\partial\Omega$.

Eigenvalue problems play a very important role in mathematical physics and engineering technology which have won extensive attention by scholars. For the second order elliptic eigenvalue problem, the standard FEMs were studied in [1–15]. Where [1–5] estimated the lower and upper bounds for the eigenvalues; [6–8] derived asymptotic expansions and extrapolations for the eigenvalues; Dari et al. [9] and the authors in [10–12] proposed posteriori error estimates and adaptive FEM, respectively; Yang and Bi [13] and Xie [14] discussed the two-grid and multigrid methods, respectively; Lin and Xie [15] introduced a multilevel correction scheme. The standard MFEMs were researched in [16–22]. In which Mercier et al. [16] analyzed the error estimates for eigenvalue and eigenvectors by mixed and hybrid FEMs; Gardini and Lin and Xie [17,18] obtained the superconvergence results of order $O(h^2)$ for the eigenvector u in L^2-norm; Lin and Xie [19] investigated the error estimates for eigenvalue and eigenvectors and derived the asymptotic expansions and extrapolation for eigenvalue; Durán et al. and Jia et al. [20,21] discussed a posteriori error estimates; Chen et al. [22] gave the two-grid method for MFEMs. However, most of the above MFEMs for elliptic eigenvalue problem focused on the conforming elements without consideration on the superconvergence and extrapolation for nonconforming elements.

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Recently, a new mixed variational form for second elliptic problem was constructed in [23,24]. Compared with the standard MFE scheme, the new mixed formulation avoids the divergence space $H(div)$ which makes the theoretical analysis simpler; and the LBB condition is automatically satisfied when the gradient of approximation space for the original variable is included in the approximation space for the auxiliary variable, which leads to easier choice for the mixed approximation spaces. Subsequently, this method was further applied to different equations (see [25–29]). For problem (1), Weng et al. [26] provided the convergence for eigenvalue and eigenpair in two-grid new MFE scheme via triangular elements, but it did not cover the superconvergence and extrapolation; Shi et al. [27] studied the new formulation with conforming MFEM, and obtained the error estimates for the eigenpair and extrapolation solution for the eigenvalue without analysis of the superconvergence and numerical experiments.

The main purpose of this article is to apply the new MFE scheme to problem (1) with nonconforming elements. The outline of this paper is organized as follows: In Sections 2 and 3, the optimal error estimates for eigenvalue and eigenpair and the approximation from below for eigenvalue are obtained. In Section 4, the superclose and superconvergence results of order $O(h^2)$ are derived for both the original variable u in broken H^1-norm and auxiliary variable $\vec{p}$ in L^2-norm based on the special property of the nonconforming EQ_1^{rot} element (when $u \in H^3(\Omega)$, the consistency error is of order $O(h^2)$ which is one order higher than that of the interpolation error $O(h)$) and the techniques of integral identity and interpolation postprocessing. In Section 5, the extrapolation solution with accuracy $O(h^3)$ for eigenvalue is received with the help of the asymptotic expansions. In Section 6, some numerical results are provided to illustrate the validity of the theoretical analysis and effectiveness of the proposed method.

2. New MFE scheme and convergence analysis

Let Ω be a polygon domain with edges parallel to the coordinate axes, T_h be a rectangular subdivision of Ω which need not to satisfy the regular condition [30]. For all $K \in T_h$, $K = [x_K - h_{x_K}, x_K + h_{x_K}] \times [y_K - h_{y_K}, y_K + h_{y_K}]$, we denote the barycenter of element K by (x_K, y_K), the four vertices and four sides are $z_i, l_i = \overline{z_i z_{i+1}}(mod 4)(i = 1, 2, 3, 4)$, respectively. $h_K = \max\{h_{x_K}, h_{y_K}\}$, $h = \max_{K \in T_h} h_K$.

The associated MFE spaces M_h (nonconforming EQ_1^{rot} element space [31–34]) and H_h (the lowest order Raviart–Thomas element space [30]) are defined by

$$\begin{cases} M_h = \{v_h : v_h|_K \in span\{1, x, y, x^2, y^2\}, \forall K \in T_h, \int_F [v_h] ds = 0, F \subset \partial K\}, \\ H_h = H_h^1 \times H_h^2 = \{\vec{q}_h = (q_h^1, q_h^2) : \vec{q}_h |_K \in span\{1, x\} \times span\{1, y\}, \forall K \in T_h\}, \end{cases}$$

where $[v_h]$ stands for the jump of v_h across the boundary F and $[v_h] = v_h$ if $F \subset \partial\Omega$.

Then we denote the norms on M_h and H_h as $\| \cdot \|_h = (\sum_{K \in T_h} | \cdot |_{1,K}^2)^{\frac{1}{2}}$ and $\| \cdot \|_0$, respectively.

Similar to [25], the corresponding interpolation operators are defined as $I_h : v \in M \to I_h v \in M_h$ and $\Pi_h : \vec{p} \in H \to \Pi_h \vec{p} \in H_h$, $I_h |_K = I_K$, $\Pi_h |_K = \Pi_K$ satisfying

$$\int_{l_i} (v - I_K v) ds = 0, \quad \int_K (v - I_K v) dxdy = 0, \quad \int_{l_i} (\vec{p} - \Pi_K \vec{p}) \cdot \vec{n}_i ds = 0, \quad \forall K \in T_h,$$

here $M = H_0^1(\Omega)$, $H = (L^2(\Omega))^2$, and $\vec{n}_i$ is the unit outer normal vector on l_i $(i = 1, 2, 3, 4)$.

Let $\vec{p} = \nabla u$, then the new mixed variational formulation for (1) is to find $(\lambda, \vec{p}, u) \in R \times H \times M$, $\| u \|_0 = 1$, such that

$$\begin{cases} (\vec{p}, \vec{q}) = (\nabla u, \vec{q}), & \forall \vec{q} \in H, \\ (\vec{p}, \nabla v) = \lambda (u, v), & \forall v \in M, \end{cases} \tag{2}$$

where $(*, \star) = \int_\Omega * \cdot \star dxdy$.

There exist eigenvalues λ_i $(i \in N)$ for Eq. (2) as follows [35]

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots, \ \lim_{i \to \infty} \lambda_i = \infty$$

and associated eigenfunctions

$$(\vec{p}_1, u_1), (\vec{p}_2, u_2), \cdots, (\vec{p}_i, u_i), \cdots,$$

where $(u_i, u_j) = \delta_{ij}$ (δ_{ij} denotes the Kronecker symbol) and $\vec{p}_i = \nabla u_i$.

The new nonconforming mixed discrete scheme for (2) is: find $(\lambda_h, \vec{p}_h, u_h) \in R \times H_h \times M_h$, $\| u_h \|_0 = 1$, such that

$$\begin{cases} (\vec{p}_h, \vec{q}_h) = (\nabla u_h, \vec{q}_h)_h, & \forall \vec{q}_h \in H_h, \\ (\vec{p}_h, \nabla v_h)_h = \lambda_h (u_h, v_h), & \forall v_h \in M_h, \end{cases} \tag{3}$$

where $(*, \star)_h = \sum_K \int_K * \cdot \star dxdy$.

Now we introduce the boundary value problem corresponding to (2): for all $f \in G$, find $(\vec{p}, u) \in H \times M$, such that

$$\begin{cases} (\vec{p}, \vec{q}) = (\nabla u, \vec{q}), & \forall \vec{q} \in H, \\ (\vec{p}, \nabla v) = (f, v), & \forall v \in M, \end{cases} \tag{4}$$

where $G = L^2(\Omega)$, and the regularity of solutions shows that

$$\| u \|_2 + \| \vec{p} \|_1 \le C \| f \|_0 . \tag{5}$$