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Logarithmic quasi-distance proximal point scalarization method for multi-objective programming

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ABSTRACT

Recently, Gregório and Oliveira developed a proximal point scalarization method (applied to multi-objective optimization problems) for an abstract strict scalar representation with a variant of the logarithmic-quadratic function of Auslender et al. as regularization. In this study, a variation of this method is proposed, using the regularization with logarithm and quasi-distance. By restricting it to a certain class of quasi-distances that are Lipschitz continuous and coercive in any of their arguments, we show that any sequence $\{(x^k, z^k)\} \subset R^n \times R^m_{++}$ generated by the method satisfies: $\{z^k\}$ is convergent; and $\{x^k\}$ is bounded and its accumulation points are weak Pareto solutions of the unconstrained multi-objective optimization problem

1. Introduction

This study considers the unconstrained multi-objective optimization problem

$$\min \{F(x) : \ x \in R^n\} \tag{1}$$

where $F = (F_1, F_2, \ldots, F_m)^T : R^n \to R^m$ is a convex mapping related to the lexicographic order generated by the cone R^m_+, i.e., for all $x, y \in R^n$ and $\lambda \in (0, 1)$, $F_i(\lambda x + (1-\lambda)y) \le \lambda F_i(x) + (1-\lambda)F_i(y), \ \ \forall i = 1, \ldots, m$. Moreover, it will be required that one of the objective functions must be coercive, i.e., there is $r \in \{1, \ldots, m\}$ such that $\lim_{\|x\| \to \infty} F_r(x) = \infty$.

The importance of multi-objective optimization can be seen from the large variety of applications presented in the literature. White [31] offers a bibliography of 504 papers describing various different applications addressing, for example, problems concerning agriculture, banking, health services, energy, industry and water. More information with regard to multi-objective optimization can be found, for example, in Section 3 and Miettinen [21].

There is a more general class of problems, known as vector optimization, that contains multi-objective optimization. See, for example, Luc [19]. On the other hand, the methods developed for this class of problem can be classified into two types: scalarization methods and extensions of nonlinear algorithms to vector optimization. Some global optimization techniques are discussed in Chinchuluun and Pardalos [10].

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The classic proximal point method to minimize a scalar convex function f: $R^n \to R$ generates a sequence $\{x^k\}$ via the iterative scheme: given a starting point $x^0 \in R^n$, then

$$x^{k+1} \in \operatorname{argmin}\{f(x) + \lambda_k \|x - x^k\|^2 : \ x \in R^n\}, \tag{2}$$

where λ_k is a sequence of real positive numbers and $\|.\|$ is the usual norm. This method was originally introduced by Martinet [20] and developed and studied by Rockafellar [27]. In recent decades the convergence analysis of the sequence $\{x^k\}$ has been extensively studied, and several extensions of the method have been developed in order to consider cases in which the function f is not convex and/or cases where the usual quadratic term in (2) is replaced by a generalized distance, e.g., Bregman distances, φ-divergences, proximal distances and quasi-distances. The papers containing these generalizations include: Chen and Teboulle [5], Iusem and Teboulle [16], Pennanen [25], Hamdi [15], Chen and Pan [6], Papa Quiroz and Oliveira [24], Moreno et al. [22] and Langenberg and Tichatschke [18].

This class of proximal point algorithms has been extended to vector optimization. The first method in this direction was the multi-objective proximal bundle method (see, Mietttinen [21]). Göpfert et al. [13] have presented a proximal point method for the scalar representation $\langle F(x), z\rangle$ with a regularization based on Bregman functions on finite dimensional spaces. Bonnel et al. [2] and Ceng and Yao [4] present a proximal algorithm with a quadratic regularization in vector form. Villacorta and Oliveira [32] also present a proximal algorithm in vector form with the regularization being a proximal distance. Gregório and Oliveira [14] present a proximal algorithm in multi-objective optimization for an abstract strict scalar representation with a variant of the logarithmic-quadratic functions of Auslender et al. [1] as regularization. Moreover, in [7] and [8], respectively, Chen presents a class of viscosity approximation methods in vector form for solving multi-objective optimization problems and a class of proximal-type methods for solving the composite multi-objective optimization problem.

We will present a brief description of the method of Gregório and Oliveira [14]. Let F: $R^n \to R^m$ be a convex application. Given the starting points $x^0 \in R^n$ and $z^0 \in R^m_{++}$ and sequences $\beta_k, \mu_k > 0, k = 0, 1, \ldots,$ the method generates a sequence $\{(x^k, z^k)\} \subset R^n \times R^m_{++}$ via the iterative scheme:

$$(x^{k+1}, z^{k+1}) \in \operatorname{argmin}\left\{f(x, z) + \beta_k H(z) + \frac{\alpha_k}{2}\|x - x^k\|^2 : \ x \in \Omega^k, z \in R^n_{++}\right\} \tag{3}$$

where $\Omega^k = \{x \in R^n : F_i(x) \le F_i(x^k)\}$, $f : R^n \times R^m_+ \to R$ satisfies the properties (P1) to (P4) (see Section 4) and $H : R^m_{++} \to R$ is such that $H(z) = \langle z/z^k - \log(z/z^k) - e, e\rangle$ where $e = (1, \ldots, 1) \in R^m$, $z/z^k = (z_1/z^k_1, \ldots, z_m/z^k_m)$ and $\log(z/z^k) = (\log(z_1/z^k_1), \ldots, \log(z_m/z^k_m))$.

We are proposing a generalization of this method considering (3) with the quasi-distance $q : R^n \times R^n \to R_+$ (see definition 2.1) in place of the Euclidean norm $\|.\|$, i.e.,

$$(x^{k+1}, z^{k+1}) \in \operatorname{argmin}\left\{f(x, z) + \beta_k H(z) + \frac{\alpha_k}{2} q^2(x, x^k) : \ x \in \Omega^k, z \in R^n_{++}\right\}. \tag{4}$$

As quasi-distances are not necessarily symmetric (see definition 2.1), they generalize the distances. Therefore, our algorithm generalizes Gregório and Oliveira's algorithm [14]. A quasi-distance is not necessarily a convex function, nor continuously differentiable, nor even a coercive function in any of its arguments. Supposing that the quasi-distance satisfies the condition (5) (see Section 2.1), then the coercivity and Lipschitz properties are recovered (see Propositions 2.1 and 2.2). This study is limited to the class of quasi-distances satisfying condition (5) that are not necessarily convex nor differentiable in any of their arguments. Accordingly, we had to proceed differently to guarantee the convergence of our method. What is more, we found a new example of a function $f : R^n \times R^m_+ \to R$ which satisfies the properties (P1) to (P4) (see proposition 4.1), which were fundamental to the convergence of our method. Gregório and Oliveira [14], drawing on the work of Fliege and Svaiter [11], supposed that the set Ω^0 is limited and established the convergence of their method. In our case, a condition of coercivity was imposed on us in only one of the objective functions, i.e., suppose that there is $r \in \{1, \ldots, m\}$ such that $\lim_{\|x\| \to \infty} F_r(x) = \infty$, and that it has as a consequence the limitation of Ω^0 (see Lemma 4.1). The importance of the limitation of the set Ω^0 is that it guarantees that the sequence $\{x^k\}$ generated by our algorithm is limited (see proposition proof 4.4 (i)).

Quasi-distances can be applied not only to computer theory (see, for example, Brattka [3] and Kunzi et al. [17]), but also to economy, for example and, more directly, to consumer choice and to utility functions (see, for example, Romanguera and Sanchis [29] and Moreno et al. [22]). Note that Moreno et al. [22] developed a proximal algorithm with quasi-distance regularization applied to non-convex and non-differentiable scalar functions, satisfying the Kurdyka–Lojasiewics inequality. And because the quasi-distance is not necessarily symmetric, they derived an economic interpretation of this algorithm, applied to habit formation. In this respect, the work of Moreno et al. encourages us, in future investigations, to seek an economic interpretation of our algorithm applied to economy-related multi-objective problems.

One important point is that our proximal algorithm and the proximal algorithm developed by Gregório and Oliveira [14] were developed in multiobjective optimization and belong to the class of proximal point scalarization methods. Meanwhile, the algorithms developed by Bonnel et al. [2], Ceng and Yao [4], and Villacorta and Oliveira [32] were developed in vector optimization and belong to the class of proximal methods in vector form. This means that the subproblems of our algorithm and Gregório and Oliveira's are problems relating to the minimization of scalar functions, while the subproblems of the other studies' algorithms are problems relating to the minimization of vector functions.

Section 2 presents concepts and results relating to quasi-distance and subdifferential theory. In Section 3, concepts and results of general multi-objective optimization theory are presented. Section 4 presents the authors' own method, where we assure the