Applied Mathematics and Computation 273 (2016) 885–896

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

High accuracy variable mesh method for nonlinear two-point boundary value problems in divergence form[☆]

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ARTICLE INFO

MSC:
65L10
65L12
65L15
65L20

Keywords:
Variable mesh
Finite difference method
Nonlinear equation
Divergence form
Two-point boundary value problems

ABSTRACT

In this article, using three grid points, we discuss variable mesh method of order three for the numerical solution of nonlinear two-point boundary value problems:

$$(p(x)y')' = f(x,y),$$
$$y(0) = A, \quad y(1) = B.$$

We first establish an identity from which general three-point finite difference approximation of various order can be obtained. We obtain a family of third order discretization using variable mesh for the differential equations. We select the free parameter available in this discretization which leads to the simplest third order method. Numerical results are provided to illustrate the proposed method and their convergence.

1. Introduction

Consider the class of two-point boundary value problem

$$\left(p(x)y'\right)' = f(x,y), \tag{1.1}$$

subjected to the boundary conditions:

$$y(0) = A, \quad y(1) = B, \tag{1.2}$$

where A and B are two constants.

We assume that for $x \in [0, 1]$, $-\infty < y < \infty$

(i) $f(x,y)$ is continuous,
(ii) $\frac{\partial f}{\partial y}$ exists and is continuous,
(iii) $\frac{\partial f}{\partial y} > 0$.

☆ This work is supported by CSIR-JRF, grant no. 09/045(1161)/2012-EMR-I.
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http://dx.doi.org/10.1016/j.amc.2015.10.030
0096-3003/

We also assume that for $x \in [0, 1]$

(iv) $p(x)$ is continuous,
(v) $p'(x)$exists and is continuous,
(vi) $p(x) > 0$.

These conditions ensure the existence of a unique solution of the boundary value problem (1.1) and (1.2) (see [6]) and we assumed to be satisfied in the problem under consideration.

The above nonlinear differential equation is an essential tool for modeling many physical situations: chemical reactions, pendulums, bending of beams & so forth. Further, nonlinear problems of type (1.1) arise in various fields of science and engineering, like quantum mechanics, fluid dynamics, turbulent interaction of waves & current, etc. In most of the cases it is extremely difficult to find analytical solution for such boundary value problems. Thus, the solution of boundary value problem is extremely important to conclude the study of physical situation. The numerical methods are therefore found to be an attractive alternate.

Some numerical and analytical methods such as shooting method [22], finite-element method [1], the Adomain decomposition method [2], homotopy perturbation method [3], homotopy analysis method [8], have been studied for obtaining approximate solutions of boundary value problems.

There is a considerable literature on numerical methods for such boundary value problems e.g. [7,10,13–15,18,21]. Keller [6] and Marchuk [4,5] have developed the second order three-point finite difference schemes for the solution of two-point boundary value problem (1.1) whereas Chawla and Katti [16] gave fourth order method. TAGE iteration methods based on cubic spline approximations on non-uniform mesh for the numerical solution of nonlinear two-point boundary value problems in non-divergence form have been studied by Mohanty et al. [19] and Mohanty and Khosla [20].

In this paper, we have discussed the variable mesh method of order three for the class of two-point boundary value problem (1.1). In Section 2, we first establish a certain identity from which general three-point finite difference approximation of various orders can be obtained. Subsequently, the method is discussed in detail. We obtain a family of third order discretizations for the differential Eq. (1.1). We select the free parameters available in these discretizations which lead to a simplest third order method. In Section 3, the third order accuracy of the method is shown. To illustrate the applicability of the proposed method several numerical examples have been solved in Section 4 and the results are presented along with their comparison with schemes which have been developed for nonlinear two-point boundary value problems in non-divergence form. Finally, the discussion on these numerical results presented in Section 5. The choice of the mesh is depend on the given problem.

2. Numerical Methods

2.1. Variable mesh discretization

We first establish certain identities from which general three-point finite difference methods of various orders can be obtained for the two-point boundary value problem (1.1).

Let N be a positive integer, $N \geq 2$, We discretize the solution region $[0, 1]$ such that $0 = x_0 < x_1 < \ldots . x_N < x_{N+1} = 1$. Let $h_k = x_k - x_{k-1},\ k = 1, 2, \ldots, N+1$, be the mesh sizes and the mesh ratio $\sigma_k = h_{k+1}/h_k > 0,\ k = 1, 2, \ldots, N$. We set $y_k = y(x_k),\ p_k = p(x_k),\ y'_k = y'(x_k)$ and $f_k = f(x_k, y_k)$.

In (1.1) we also set

$$z(x) = p(x)y'$$

and

$$z_k = p_k y'_k, \quad k = 0, 1, \ldots, N+1$$

Integrating (1.1) from x_k to x, we obtain

$$z(x) = z_k + \int_{x_k}^{x} f(t)dt, \tag{2.1.1}$$

where we have set

$$f(t) = f(t, y(t)).$$

Dividing (2.1.1) by $p(x)$, integrating from $x_k$to x_{k+1}, we obtain

$$\frac{y_{k+1} - y_k}{J_k(h_k)} = \sigma_k h_k (z_k + \sigma_k h_k I_k(h_k)), \tag{2.1.2}$$

where we have set

$$P(x) = 1/p(x),$$

$$J_k(h_k) = \int_0^1 P(x_k + \sigma_k h_k u)du, \tag{2.1.3}$$