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A new family of optimal eighth order methods with dynamics for nonlinear equations

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A R T I C L E I N F O

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A B S T R A C T

We propose a simple yet efficient family of three-point iterative methods for solving nonlinear equations. Each method of the family requires four evaluations, namely three functions and one derivative, per full iteration and possesses eighth order of convergence. Thus, the family is optimal in the sense of Kung–Traub conjecture and has the efficiency index 1.682 which is better than that of Newton's and many other higher order methods. Various numerical examples are considered to check the performance and to verify the theoretical results. Computational results including the elapsed CPU-time, confirm the efficient and robust character of proposed technique. Moreover, the presented basins of attraction also confirm better performance of the methods as compared to other established methods in literature.

1. Introduction

The construction of fixed point iterative methods for solving the nonlinear equation $f(x) = 0$ is an important and interesting task in numerical analysis and many applied scientific disciplines. A great importance of this subject has led to the development of many iterative methods. Although iterative methods were extensively studied in Traub's book [1] and some books and papers published in the 1960s, 70s and 80s (see, e.g., [2–9]). The interest for these methods has renewed in recent years due to the rapid development of digital computers, advanced computer arithmetics (multi-precision arithmetic and interval arithmetic) and symbolic computation.

Traub [1] has divided iterative methods into two classes, viz. one-point iterative methods and multipoint iterative methods. The important aspects related to these classes of methods are order of convergence and computational efficiency. Order of convergence shows the speed with which a given sequence of iterates converges to the root while the computational efficiency concerns with the economy of the entire process and is defined by $E = p^{1/n}$ (see [8]), where p is the order of convergence and n is the total number of function evaluations required per iteration. Investigation of one-point iteration methods has demonstrated theoretical restrictions on the order and efficiency of these methods (see [1]). However, Kung and Traub [6] have conjectured that multipoint iteration methods without memory based on n evaluations have optimal order 2^{n-1}. For example, with three function evaluations a two-point method of optimal fourth order convergence can be constructed (see [3–5,8,10–13]) and with four function evaluations a three-point method of optimal eighth order convergence can be developed (see [7,14–26]). A more extensive list of references as well as a survey on progress made on the class of multipoint methods may be found in the recent book by Petković et al. [27].

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Motivated by optimization considerations, we develop a family of three-point iterative methods which has simple structure and possesses optimal eighth order convergence. The scheme is composed of three steps of which the first two steps consist of any fourth order method with the base as well-known Newton's iteration. Rest of the paper is outlined as follows. In Section 2 the new family is developed and its convergence analysis is discussed. The theoretical results proved in Section 2 are verified in Section 3 by considering various numerical examples. A comparison of the new methods with the existing methods is also performed in this section. In Section 4, the basins of attractors for the new proposed methods and some existing eighth order methods are presented. Concluding remarks are given in Section 5.

2. The family of methods

Let us consider iterative methods for solving the nonlinear equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D. Let α be a simple zero of the function $f(x)$ and x_0 be an initial approximation to α. We begin with the following simple three-point scheme

$$\begin{cases} y_i = x_i - \dfrac{f(x_i)}{f'(x_i)}, \\ z_i = \phi_4(x_i, y_i), \\ x_{i+1} = z_i - \dfrac{f[z_i, y_i]}{f[z_i, x_i]} \dfrac{f(z_i)}{\mu f[z_i, x_i] + \nu f[z_i, y_i]}, \quad i = 0,\ 1,\ 2, \ldots. \end{cases} \tag{1}$$

Here $\phi_4(x_i, y_i)$ is any two-point optimal fourth order scheme with the base as Newton's iteration y_i, $f[r, s] = \frac{f(r)-f(s)}{r-s}$ is Newton's first order divided difference, and μ and ν are two arbitrary constants to be determined.

Through the following theorem we analyze the behavior of scheme (1):

Theorem 1. *Let the function $f(x)$ be sufficiently differentiable in a neighborhood of its simple zero α and $\phi_4(x_i, y_i)$ is an optimal fourth order method which satisfies*

$$z_i - \alpha = \sum_{j=4}^{8} B_{j-4} e_i^j + O(e_i^9), \tag{2}$$

where $B_0 \neq 0$ and $e_i = x_i - \alpha$. If the initial approximation x_0 is sufficiently close to α, then the order of convergence of (1) is at least 8, provided $\mu = -1$ and $\nu = 2$.

Proof. Let $e_{y_i} = y_i - \alpha$ and $e_{z_i} = z_i - \alpha$ be the errors in the ith iteration. Using Taylor's expansion of $f(x_i)$ about α and taking into account that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, we have

$$f(x_i) = f'(\alpha)\left[\sum_{j=1}^{8} A_j e_i^j + O(e_i^9)\right], \quad A_j = (1/j!) f^{(j)}(\alpha)/f'(\alpha),\ \ j = 1, 2, 3, \ldots. \tag{3}$$

Also,

$$f'(x_i) = f'(\alpha)\left[\sum_{j=1}^{8} jA_j e_i^{j-1} + O(e_i^8)\right]. \tag{4}$$

Substitution of (3) and (4) in the first step of (1) yields

$$e_{y_i} = \sum_{j=2}^{8} C_{j-2} e_i^j, \tag{5}$$

where

$$\begin{aligned} C_0 &= A_2,\ C_1 = -2(A_2^2 - A_3),\ C_2 = 4A_2^3 - 7A_2A_3 + 3A_4,\ C_3 = -2(4A_2^4 - 10A_2^2A_3 + 3A_3^2 + 5A_2A_4 - 2A_5), \\ C_4 &= 16A_2^5 - 52A_2^3A_3 + 33A_2A_3^2 + 28A_2^2A_4 - 17A_3A_4 - 13A_2A_5 + 5A_6, \\ C_5 &= -2(16A_2^6 - 64A_2^4A_3 + 63A_2^2A_3^2 - 9A_3^3 + 36A_2^3A_4 - 46A_2A_3A_4 + 6A_4^2 - 18A_2^2A_5 + 11A_3A_5 + 8A_2A_6 - 3A_7), \\ C_6 &= 64A_2^7 - 304A_2^5A_3 + 408A_2^3A_3^2 - 135A_2A_3^3 + 176A_2^4A_4 - 348A_2^2A_3A_4 + 75A_3^2A_4 + 64A_2A_4^2 - 92A_2^3A_5 + 118A_2A_3A_5 \\ &\quad - 31A_4A_5 + 44A_2^2A_6 - 27A_3A_6 - 19A_2A_7 + 7A_8. \end{aligned}$$

Expanding $f(y_i)$ about α, we obtain

$$f(y_i) = f'(\alpha)\left[\sum_{j=1}^{4} A_j e_{y_i}^j + O(e_{y_i}^5)\right]. \tag{6}$$

Also, expansion of $f(z_i)$ about α yields

$$f(z_i) = f'(\alpha)[e_{z_i} + A_2 e_{z_i}^2 + O(e_{z_i}^3)]. \tag{7}$$