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# Analysis of codimension 2 bifurcations for high-dimensional discrete systems using symbolic computation methods

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ABSTRACT

This article reports an algebraic criterion of the eigenvalue assignment, transversality condition and non-resonance condition for fold-N–S bifurcations. By means of symbolic computation methods, we propose an algorithmic approach for systematically analyzing codimension 2 bifurcations for high-dimensional discrete systems. The effectiveness of the proposed symbolic approach is verified by experiments. In particular, the flip- and fold-N–S bifurcations of a five-dimensional discrete dynamical system with a washout-filter feedback controller are analyzed.



## 1. Introduction

Many biological phenomena and control problems may be modeled mathematically by continuous or discrete dynamical systems (see, e.g., [10,31]). Most of such systems are non-linear, and their explicit analytic solutions are difficult to find in general. Therefore in order to understand the phenomena or problems described by dynamical systems, it is important to study their behaviors such as stability, bifurcations, limit cycles, and chaos qualitatively. For the stability analysis, one usually uses the technique of linearization, but this method may fail at bifurcation points. Around such points, the dynamical behaviors of the original systems may be quite different from those of the linearized ones. That is to say, small perturbations of the (bifurcation) parameters may lead to a sudden "qualitative" change to the properties of the dynamical systems like the loss of stability. Bifurcations, which are tightly related to the occurrence of limit cycles or chaos, are of significance in practical problems, e.g., [4,15]. Furthermore, the analysis of bifurcations is a highly nontrivial task. It is usually difficult to apply numerical methods to this analysis, for one can hardly determine the conditions on the parameters for the occurrence of bifurcations by using numerical simulations [5]. Therefore it is of importance and our interest to study approaches for the analysis of bifurcations based on the newly developed methods for exact computations.

Similar to the continuous systems, there are three principal codimension 1 bifurcations in the discrete case: the fold, flip, and Neimark–Sacker (N–S for short hereafter) bifurcations [21]. In high-dimensional cases, the interaction between two bifurcations may result in a new category of bifurcations, which are the so-called codimension 2 (codim 2 for short hereafter) bifurcations. There are many different specific codim 2 bifurcations for high-dimensional discrete systems. For example, 11 codim 2 bifurcations have been introduced in [17]. In this paper, however, we mainly focus on two of them, namely the flip-N–S and fold-N–S bifurcations.

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We are concerned with the following first-order autonomous discrete difference equations (high-order systems can be transformed into first-order ones)

$$\begin{cases} x_1(t+1) = f_1(\mu_1(t), \dots, \mu_m(t), x_1(t), \dots, x_n(t)), \\ \quad\vdots \\ x_n(t+1) = f_n(\mu_1(t), \dots, \mu_m(t), x_1(t), \dots, x_n(t)), \end{cases} \tag{1}$$

where $\mu_1, \dots, \mu_m$ are parameters dependent on $t$, $x_1, \dots, x_n$ are variables, and $f_1, \dots, f_n$ are mappings from $\mathcal{K}^{m+n}$ to $\mathcal{K}$ with $\mathcal{K}$ the base field. For the sake of simplicity, we write $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$, $\boldsymbol{x} = (x_1, \dots, x_n)$, and $f_i : \mathcal{K}^{m+n} \to \mathcal{K}$.

Discrete dynamical systems in the form of (1) may serve as the underlying mathematical model for many practical problems, e.g., [2]. For such systems, Wen and Xu [30] proposed an algebraic criterion for the occurrence of double N–S bifurcations in 5- and 6- dimensional systems and numerically verified a set of parameter values under which a double N–S bifurcation appears. Furthermore, a criterion of flip-N–S bifurcations for $n$-dimensional discrete systems was proved in [35] and numerical computations on a set of concrete parameter values were performed to verify the occurrence of bifurcations. The computations in these two papers are all based on numerical methods and only several concrete parameter values can be verified. Symbolic methods (with exact computations), including Gröbner bases [6,9], triangular decomposition [27,32], quantifier elimination [8,14], real solution classification [34] and discriminant varieties [20], have also been used to analyze discrete dynamical systems in the form of (1) as regards their three types of codim 1 bifurcations [21].

In this paper we study algebraically and symbolically the conditions on the parameters for the discrete dynamical systems in the form of (1) to have (prescribed numbers of) flip- or fold-N–S bifurcations. The main contributions include: (a) Theorem 2, the main theorem of the paper which states a necessary and sufficient algebraic condition on the occurrence of fold-N–S bifurcations (in Section 2.2). This condition is derived with detailed analyses on the eigenvalue assignment, the transversality condition and the non-resonance condition of fold-N–S bifurcations. To our best knowledge, the derived condition is likely to be the first algebraic one for fold-N–S bifurcations. (b) The specific semi-algebraic problem to solve for the analysis of flip- and fold-N–S codim 2 bifurcations (in Section 3). With the aid of the main theorem, we reduce the analysis of codim 2 bifurcations to the solution of a semi-algebraic system, which is a well investigated subject in Symbolic Computation. (c) A systematical approach for rigorously analyzing the conditions on the parameters under which a certain type of codim 2 bifurcation occurs by using symbolic computation methods (in Section 4). (d) Experiments on the analysis of flip- and fold-N–S bifurcations for specific systems which are used to model practical problems (in Section 5). The experimental results verify the effectiveness of the proposed approach.

## 2. Algebraic criteria for codimension 2 bifurcations

For the study on the occurrence of codim 2 bifurcations at steady states, one needs to study three conditions: eigenvalue assignment, transversality condition, and non-resonance condition [16]. The first two guarantee the existence of a bifurcation, and the third is for the type of bifurcation.

Now let us restricted to a class of discrete dynamical systems in the form (1) with

$$f_i(\boldsymbol{\mu}, \boldsymbol{x}) = \frac{P_i(\boldsymbol{\mu}, \boldsymbol{x})}{Q_i(\boldsymbol{\mu}, \boldsymbol{x})} \quad (i = 1, \dots, n) \tag{2}$$

as rational functions, where $P_i$ and $Q_i$ are polynomials in $\boldsymbol{\mu}$ and $\boldsymbol{x}$ with coefficients from $\mathbb{R}$, the real number field. The main aims of this paper are to give algebraic criteria of two kinds of codim 2 bifurcations: flip-N–S bifurcations and fold-N–S bifurcations for system (1)+(2) so that the computation of analytical expressions of eigenvalues can be avoided, and to analyze these bifurcations by using exact symbolic computations.

In this section, we first recall the definition of flip-N–S bifurcations and an algebraic criterion for the occurrence of such kind of codimension 2 bifurcations proposed in [35]. Then, based on this work, we deduce the critical algebraic conditions for the occurrence of fold-N–S bifurcations.

### *2.1. Flip-N–S bifurcation*

Suppose that $\boldsymbol{x} = \bar{\boldsymbol{x}}$ represents a steady state of system (1). Let $\boldsymbol{\mu} = \bar{\boldsymbol{\mu}}$ denote the parameters with which the bifurcation may occur and $\bar{J} = J(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}})$ denote the Jacobian matrix of system (1) at $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{\mu}}$.

Firstly, the eigenvalue assignment of flip-N-S bifurcations as mentioned above means that $\bar{J}$ has a pair of complex conjugate eigenvalues $\lambda_1(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}})$ and $\bar{\lambda}_1(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}})$ simultaneously lying on the unit circle, namely $|\lambda_1(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}})| = 1$, one real eigenvalue $\lambda_3(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}}) = -1$, and the others (if they exist) lying inside the unit circle, namely $|\lambda_i(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}})| < 1$ for $i = 4, \dots, n$. Secondly, the transversality condition means that the pair of complex conjugate eigenvalues and the real eigenvalue should cross the unit circle at a nonzero speed, and this condition can be represented by $\frac{\partial |\lambda_i(\boldsymbol{\mu}, \boldsymbol{x})|}{\partial \boldsymbol{\mu}}|_{\boldsymbol{\mu}=\bar{\boldsymbol{\mu}}} \neq 0, i = 1, 3$. Lastly, the non-resonance condition can be represented by $\lambda_1^l(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{x}}) \neq 1, l = 3, 4, 5, \dots$.

For $n$-dimensional discrete dynamical systems, Yao in [35] proposed a criterion for detecting the flip-N–S bifurcation without computing these eigenvalues. This criterion is derived from the coefficients of the characteristic polynomial of the system, instead of the analytical expression of eigenvalues of Jacobian matrix of the system. The criterion consists of a series of equations, inequalities and inequations as described in Lemma 1.