Applied Mathematics and Computation 273 (2016) 957–968

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On a family of Weierstrass-type root-finding methods with accelerated convergence

Petko D. Proinov*, Maria T. Vasileva

Faculty of Mathematics and Informatics, University of Plovdiv, Plovdiv 4000, Bulgaria

ARTICLE INFO

MSC:
65H04
12Y05

Keywords:
Simultaneous methods
Weierstrass method
Accelerated convergence
Local convergence
Semilocal convergence
Error estimates

ABSTRACT

Kyurkchiev and Andreev (1985) constructed an infinite sequence of Weierstrass-type iterative methods for approximating all zeros of a polynomial simultaneously. The first member of this sequence of iterative methods is the famous method of Weierstrass (1891) and the second one is the method of Nourein (1977). For a given integer $N \geq 1$, the Nth method of this family has the order of convergence $N+1$. Currently in the literature, there are only local convergence results for these methods. The main purpose of this paper is to present semilocal convergence results for the Weierstrass-type methods under computationally verifiable initial conditions and with computationally verifiable a posteriori error estimates.

1. Introduction and preliminaries

Throughout this paper $(\mathbb{K}, |\cdot|)$ denotes an algebraically closed normed field and $\mathbb{K}[z]$ denotes the ring of polynomials (in one variable) over $\mathbb{K}$. We endow the vector space $\mathbb{K}^n$ with the p-norm $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for some $1 \leq p \leq \infty$, and we equip $(\mathbb{R}^n, \|\cdot\|_p)$ with coordinate-wise ordering $\preceq$ defined by

$$x \preceq y \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for each} \quad i = 1, \dots, n. \tag{1.1}$$

Then $(\mathbb{R}^n, \|\cdot\|_p)$ is a solid vector space. Also we define a cone norm $\|\cdot\|$ in $\mathbb{K}^n$ with values in $\mathbb{R}^n$ by

$$\|x\| = (|x_1|, \dots, |x_n|). \tag{1.2}$$

Then $(\mathbb{K}^n, \|\cdot\|)$ is a cone normed space over $\mathbb{R}^n$ (see, e.g., Proinov [9]).

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. A vector $\xi \in \mathbb{K}^n$ is called a root-vector of f if $f(z) = a_0 \prod_{i=1}^n (z - \xi_i)$ for all $z \in \mathbb{K}$, where $a_0 \in \mathbb{K}$.

In 1891, Weierstrass [18] published his famous iterative method for simultaneous computation of all zeros of f. The *Weierstrass method* is defined by the following iteration:

$$x^{(k+1)} = x^{(k)} - W_f(x^{(k)}), \quad k = 0, 1, 2, \dots, \tag{1.3}$$

where the operator $W_f : \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by

$$W_f(x) = (W_1(x), \dots, W_1(x)) \quad \text{with} \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)} \quad (i = 1, \dots, n), \tag{1.4}$$

* Corresponding author. Tel.: +359 888802163.
E-mail addresses: proinov@uni-plovdiv.bg (P.D. Proinov), mariavasileva@uni-plovdiv.bg (M.T. Vasileva).

http://dx.doi.org/10.1016/j.amc.2015.10.048
0096-3003/

where $a_0 \in \mathbb{K}$ is the leading coefficient of f and the domain $\mathcal{D}$ of W_f is the set of all vectors in $\mathbb{K}^n$ with distinct components. The Weierstrass method (1.3) has second-order of convergence provided that all zeros of f are simple. Other iterative methods for simultaneous finding polynomial zeros can be found in the books [4,5,8,15] and the references therein.

In 1985, Kyurkchiev and Andreev [3] introduced a sequence of iterative methods for approximating all zeros of a polynomial simultaneously. The first member of their family of iterative methods is the Weierstrass method (1.3) and the second one is the method of Nourein [7].

Before we present Kyurkchiev and Andreev's family of iterative methods, we give some notations which will be used throughout the paper. We define the binary relation # on $\mathbb{K}^n$ by

$$x \# y \quad \Leftrightarrow \quad x_i \neq y_j \text{ for all } i, j \in I_n \text{ with } i \neq j. \tag{1.5}$$

Here and throughout this paper, we denote by I_n the set of indices $1, \dots, n$. For two vectors $x \in \mathbb{K}^n$ and $y \in \mathbb{R}^n$ we define in $\mathbb{R}^n$ the vector

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \dots, \frac{|x_n|}{y_n} \right)$$

provided that y has only nonzero components. We define the function $d : \mathbb{K}^n \to \mathbb{R}^n$ by

$$d(x) = (d_1(x), \dots, d_n(x)) \quad \text{with} \quad d_i(x) = \min_{j \neq i} |x_i - x_j| \quad (i = 1, \dots, n).$$

In the sequel, for a given vector x in $\mathbb{K}^n$, x_i always denotes the ith component of x. In particular, if F is a map with values in $\mathbb{K}^n$, then $F_i(x)$ denotes the ith component of the vector $F(x)$.

Definition 1.1. Suppose $f \in \mathbb{K}[z]$ is a polynomial of degree $n \geq 2$. We define the sequence $(T^{(N)})_{N=0}^{\infty}$ of functions $T^{(N)} : D_N \subset \mathbb{K}^n \to \mathbb{K}^n$ recursively by setting $T^{(0)}(x) = x$ and

$$T_i^{(N+1)}(x) = x_i - \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - T_j^{(N)}(x))} \quad (i = 1, \dots, n), \tag{1.6}$$

where the sequence of the domains D_N is also defined recursively by setting $D_0 = \mathbb{K}^n$ and

$$D_{N+1} = \{x \in D_N : x \# T^{(N)}(x)\}. \tag{1.7}$$

Let $N \in \mathbb{N}$ be fixed. Then the Nth method of Kyurkchiev–Andreev's family can be defined by the following fixed point iteration:

$$x^{(k+1)} = T^{(N)}(x^{(k)}), \quad k = 0, 1, 2, \dots. \tag{1.8}$$

Currently in the literature, there are only local convergence results for the Weierstrass-type methods (1.8) (see [3,13]). In this paper, we present semilocal convergence results for the Weierstrass-type methods under computationally verifiable initial conditions and with computationally verifiable a posteriori error estimates. These results are obtained by using some results of [10] and [11].

The paper is structured as follows: in Section 2, we obtain new local convergence results (Theorem 2.11, Corollaries 2.12 and 2.13) for the Weierstrass-type methods (1.8). In the case $N = 1$ (Weierstrass method) and $p = \infty$ the main result of this section reduces to a result of Proinov [10, Theorem 7.3]. In Section 3, we present our semilocal convergence results (Theorems 3.2, 3.5, Corollaries 3.3 and 3.6) for the Weierstrass-type methods (1.8). Note that these results are based on the local convergence results obtained in the previous section. In Section 4, we provide three numerical examples to show the applicability of our semilocal convergence results.

Throughout this paper, we follow the terminology of [10]. In particular we refer to this paper for the definition of the following notions: quasi-homogeneous function of degree $r \geq 0$; gauge function of order $r \geq 1$; function of initial conditions of a map; initial point of a map; iterated contraction at a point.

2. Local convergence analysis of the Weierstrass-type methods

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. In this section, we study the convergence of the Weierstrass-type methods (1.8) with respect to the function of initial conditions $E : \mathcal{D} \to \mathbb{R}_+$ defined by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \tag{2.1}$$

for some $1 \leq p \leq \infty$. We define the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\Psi(t) = (1 + 2t) \left(1 + \frac{t}{(n-1)^p} \right)^{n-1}. \tag{2.2}$$