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Rational cubic clipping with linear complexity for computing roots of polynomials

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ABSTRACT

Many problems in computer aided geometric design and computer graphics can be turned into a root-finding problem of polynomial equations. Among various clipping methods, the ones based on the Bernstein–Bézier form have good numerical stability. One of such clipping methods is the k-clipping method, where $k = 2, 3$ and often called a cubic clipping method when $k = 3$. It utilizes $O(n^2)$ time to find two polynomials of degree k bounding the given polynomial $f(t)$ of degree n, and achieves a convergence rate of $k + 1$ for a single root. The roots of the bounding polynomials of degree k are then used for bounding the roots of $f(t)$. This paper presents a rational cubic clipping method for finding two bounding cubics within $O(n)$ time, which can achieve a higher convergence rate 5 than that of 4 of the previous cubic clipping method. When the bounding cubics are obtained, the remaining operations are the same as those of previous cubic clipping method. Numerical examples show the efficiency and the convergence rate of the new method.

1. Introduction

Many problems in computer aided geometric design and computer graphics can be turned into a root-finding problem of polynomial equations, such as robotics [15], computer aided design and manufacturing [8,19], including curve/surface intersection [7,13,18,19], point projection [2], collision detection [5,12], and bisectors/medial axes computation [8]. In principle, a system of polynomial equations of multiple variables can also be turned into a univariate polynomial equation by using the resultant theory. This paper discusses the root-finding problem of a univaraite polynomial equation within an interval.

In many references, the given polynomial $f(t)$ of degree n is written into its power series [10,14,17,20]. However, the Bernstein–Bézier form of $f(t)$ has good stability with respect to perturbations of the coefficients [9,11]. Several methods based on the Bernstein–Bézier form are developed [1,13,16,21], which are proved to achieve good numerical stability. Note that the number of zeros of a Bézier function is less or equal to that of its control polygon, the method in [16] utilizes the corresponding control polygon to approximation $f(t)$, in which the zeros of the control polygon is used to approximate the zeros of $f(t)$ from one side. The method in [16] achieves the convergence rate 2 for a simple root. In principle, a B-spline (or Bézier) curve is bounded by the convex hull of its control polygon, the corresponding roots are then bounded by the roots of the convex hull, i.e., the roots are bounded from two sides. The corresponding approximation order of the convex hull is 2. Comparing with the method in [16], the k-clipping method in [1,13] bounds the zeros of $f(t)$ by using the zeros of two bounding polynomials of degree k, which

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achieves a higher approximation order $k+1$, where k is usually set as 2 or 3. Recently, Chen et. al presented a planar quadratic clipping method by using rational quadratic polynomials to approximate $(t, f(t))$ in $\mathbb{R}^2$ space, which achieves the convergence rate 4 for a simple root [3]. In principle, for a case that $(t, f(t))$ is convex within the given interval, the rational quadratic clipping method in [4] can achieve the convergence rate 5 for a single root. However, in case when the curve $(t, f(t))$ is not convex within $[a, b]$, the denominators of the rational quadratic polynomials for bounding $f(t)$ may have one or more zeros within $[a, b]$, which leads to a bad approximation effect between $f(t)$ and its bounding polynomials. In principle, the above clipping methods have mainly two steps for each clipping process, i.e., one is to compute the two bounding polynomials whose computation time is dominant, and the other is to obtain the resulting subintervals based on the roots of the bounding polynomials within $O(n)$ time. In the worst case, the above methods need $O(n^2)$ time to compute the two bounding polynomials during each clipping process, and the total computational complexity is $O(n^2)$.

This paper presents a fast cubic clipping method of linear computational complexity. The main contribution is to directly construct the two bounding cubics within $O(n)$ time. It is proved that the bounding cubics achieve the approximation order 4 to $f(t)$, which is the same as that of previous cubic clipping method in [13]. Once the bounding cubics are obtained, the remaining operations are the same as that of the cubic clipping method in [13]. Numerical examples confirms higher computational efficiency of the new method.

The remainder of this paper is organized as follows. Section 2 provides an outline of the proposed clipping method and discusses two key issues of a general clipping method. Section 3 presents the method for constructing two bounding cubics within a linear time. Section 4 illustrates the algorithm with several numerical examples and other related discussions. Conclusions are drawn at the end of this paper.

2. Clipping methods

2.1. Outline of clipping methods

The basic idea of clipping methods is to iteratively clip the parts of an interval containing no roots of the given polynomial $f(t)$. In each clipping process, there are mainly two steps:

(1) The first step is to compute two bounding polynomials, which is a key issue of a clipping method.
(2) The second step is to divide the given interval into several subintervals by using the roots of the bounding polynomials, and clip the subintervals containing no roots of $f(t)$.

In principle, different clipping methods may obtain different bounding polynomials. Once two bounding polynomials are obtained such that $g_1(t) \le f(t) \le g_2(t)$. The remaining work is to compute the roots of $g_i(t)$, to rearrange the roots in order such that $a = t_0^\star < t_1^\star < t_2^\star < \cdots < t_l^\star = b$, and to remove the subintervals containing no roots of $f(t)$. Solving a polynomial equation of degree 3 or less seems to be trivial. The details of judging whether or not a subinterval Λ_i contains one or more roots of $f(t)$ are as follows. It can be divided into three cases:

- Case 1: If $g_1(t) > 0$ for all $t \in \Lambda_i$, Λ_i can be removed.
  From the assumption, we have that $f(t) \ge g_1(t) > 0$, for all $t \in \Lambda_i$, which means that Λ_i contains no root of $f(t)$ and can thus be removed.
- Case 2: If $g_2(t) < 0$ for all $t \in \Lambda_i$, Λ_i can be removed.
  From the assumption, we have that $f(t) \le g_2(t) < 0$, for all $t \in \Lambda_i$, which means that Λ_i contains no root of $f(t)$ and can also be removed.
- Case 3: $g_1(t) \le 0$ and $g_2(t) \ge 0$, Λ_i is kept for further dealing.

2.2. The analysis of convergence rate

The analysis of convergence rate is another key issue of a clipping method, which depends on the approximation order of the bounding polynomials. We have the following theorem.

Theorem 1. *Suppose that the two bounding polynomials $g_i(t)$, $i = 1, 2$, achieve the approximation order m to $f(t)$ within interval $[a, b]$ whose length is small enough for satisfying Eq. (2), then the corresponding convergence rate for a root $t^\star$ of multiplicity k is $\frac{m}{k}$.*

Proof. From the assumption that $t^\star$ has multiplicity k, we have that

$$f^{(j)}(t^\star) = 0, \quad \text{for} \quad j = 0, 1, \ldots, k-1, \quad \text{and} \quad f^{(k)}(t^\star) \neq 0. \tag{1}$$

Combining Taylor's expansion with Eq. (1), there exists a small enough $\eta > 0$ and $\xi_1(t)$ such that

$$\begin{aligned} |f(t) - f(t^\star)| &= \left| \frac{f^{(k)}(t^\star)}{k!}(t-t^\star)^k + \frac{f^{(k+1)}(\xi_1(t))}{(k+1)!}(t-t^\star)^{k+1} \right| \\ &> \left| \frac{1}{2}\frac{f^{(k)}(t^\star)}{k!}(t-t^\star)^k \right|, \quad \forall\, t \in [t^\star - \eta, t^\star + \eta]. \end{aligned} \tag{2}$$