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Stable spectral collocation solutions to a class of Benjamin Bona Mahony initial value problems

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ABSTRACT

We are concerned with stable spectral collocation solutions to non-periodic Benjamin Bona Mahony (BBM), modified BBM and Benjamin Bona Mahony-Burgers (BBM-B) initial value problems on the real axis. The spectral collocation is based alternatively on the scaled Hermite and sinc functions. In order to march in time we use several one step and linear multistep finite difference schemes such that the method of lines (MoL) involved is stable in sense of Lax. The method based on Hermite functions ensures the correct behavior of the solutions at large spatial distances and in long time periods. In order to prove the stability we use the pseudospectra of the linearized spatial discretization operators. The extent at which the energy integral of BBM model is conserved over time is analyzed for Hermite collocation along with various finite difference schemes. This analysis has been fairly useful in optimizing the scaling parameter. The effectiveness of our approach has been confirmed by some numerical experiments.

1. Introduction

The purpose of this paper is the analysis and practical development of the method of lines (MoL for short) based on spectral collocation and various finite difference schemes as an efficient tool to solve Cauchy's problem attached to a class of Benjamin Bona Mahony (BBM for short) equations. The most representative equation reads

$$u_t + u_x - \alpha u_{xx} + u^n\, u_x - u_{xxt} = 0, \quad 0 \le \alpha \le 1, \quad n \in \mathbb{N}, \tag{1}$$

whose solution $u(x, t)$ is considered in a class of non-periodic functions defined on $-\infty < x < +\infty$ and $t \ge 0$.

For $\alpha = 0$ and $n = 1$ Eq. (1) is called Benjamin Bona Mahony or the *regularized long-wave equation*. It has been introduced by the above mentioned mathematicians in the early '70 (see [1]) as a counterpart of the well known Korteweg-de Vries (KdV) equation

$$u_t + u_x + u\, u_x + u_{xxx} = 0. \tag{2}$$

When $\alpha = 0$ and $n > 1$ Eq. (1) will be called modified BBM equation and for $\alpha > 0$ and $n = 1$ we will refer at (1) as to the Benjamin Bona Mahony-Burgers (BBM-B for short) one.

In (1), as well as in the formulation of (2), $u = u(x, t)$ represents the vertical displacement of the surface of the liquid (ideal fluid) from its equilibrium position, t is the time and x is the horizontal coordinate which increases in the direction of waves

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propagation. Both equations are in dimensionless form with the length scale taken to be the unperturbed depth h of the liquid and time scale be $(h/g)^{1/2}$; g is the gravity constant.

The authors of [1] compare in various ways their regularized long-wave equation with KdV and argue that the latter has some shortcomings which makes that an unsuitable posed model for long waves (see also [2]). We do not want to get involved into the details of this debate but we observe that the BBM model has received less attention from numerical point of view than the KdV one.

One of the purposes of this work is to try to fill this gap providing reasonable numerical wave solutions without an artificial (empirical) truncation of the domain or periodicity assumptions.

For a model equation for long waves, periodic solutions have been obtained in the early era of modern computational mathematics by Galerkin method in [19]. On a finite interval, an initial-(Dirichlet) boundary value problem attached to BBM has been solved by the Galerkin method in [13] and by a Crank–Nicolson-type finite difference method in [14].

A kind of Hermite spectral method has been used in [11] in order to solve KdV, a modified KdV and the nonlinear Schroedinger equations. The so called *KdV solitons of Boyd* have been detected this way. However, in [3] it is observed that the Hermite functions are the exact eigenfunctions of quantum mechanics harmonic oscillator and are the *asymptotic* eigenfunctions for Mathieu's equation, the prolate spheroidal wave equation, the associated Legendre equation, and Laplace's Tidal equation. This close connection with the physics makes Hermite functions a fairly natural choice of basis functions for various fields of science and engineering.

In a recent laborious study [4] the authors extend the framework of the finite volume methods to dispersive water wave models, in particular to Boussinesq type systems.

In this context, we focus in this paper on the mathematical aspects of the spectral collocation methods, mainly based on Hermite functions, as an efficient tool in modeling the dispersion of long waves. Along with some implicit time schemes, the involved MoL, accurately conserves a quadratic invariant of the flow and resolves correctly the asymptotic behavior of the solutions at large distances.

Specifically, we will be concerned in the first instance with Cauchy's problem

$$\begin{cases} u_t + u_x + u\,u_x - u_{xxt} = 0, & -\infty < x < +\infty, \quad t \geq 0, \\ u(x,0) = g(x), \end{cases} \tag{3}$$

where the initial data $g(x)$ satisfies the conditions

1. $g \in W_2^1(\mathbb{R}) \cap C^2(\mathbb{R})$,
2. $\int_{-\infty}^{\infty} (g^2 + g_x^2)dx := E_0 < \infty$.

Then the problem (3) has a solution $u \in \mathcal{C}_{\infty}^{2,\infty}$. At any $t \in [0, \infty)$ the solution u and its derivative u_x along with all their derivatives with respect to t are asymptotically null i.e.,

$$u, \quad u_x, \quad u_t, \quad u_{xt}, \ldots \to 0 \quad as \quad x \to \pm\infty. \tag{4}$$

Additionally, E_0 is conserved over time, i.e., $E(u) = E_0$ where

$$E(u) := \int_{-\infty}^{\infty} \left(u^2 + u_x^2\right)dx. \tag{5}$$

With respect to the first condition on the initial data g, the authors of [1] observe that in fact an element g of $L_2(\mathbb{R}) \cap C^1(\mathbb{R})$ and its first derivative g_x, both converge to 0 at $\pm\infty$. The functional E introduced with (5) is called *energy integral* although it is not necessarily identifiable with the energy in the original physical problems.

In order to describe the smoothness of solution the authors quoted above introduce the space $\mathcal{C}_T$ of continuous functions $u(x, t)$ on $\mathbb{R} \times [0, T)$ equipped with the maximum norm and its subspace $\mathcal{C}_T^{l,m}$ of functions u such that $\partial_x^i \partial_t^l u \in \mathcal{C}_T$ for $0 \leq i \leq l$ and $0 \leq j \leq m$ equipped with the subsequent norm. The space $W_2^1(\mathbb{R})$ is the usual Sobolev space of L_2 functions over $\mathbb{R}$ with generalized first derivative which is also L_2 function.

We will heavily rely on this analytical result. Moreover this fundamental result on the existence, uniqueness and asymptotic behavior of *non-periodic solutions* to (3) has suggested a *collocation approach* of the spatial dependence using Hermite (HC for short) as well as sinc functions (SC for short). In this context fairly accurate differentiation matrices corresponding to these functions are needed. They are available in [5] and in the seminal paper [20].

Actually, the HC method is used in the same spirit in which the Laguerre collocation (LC) has been successfully used in our previous papers [6] and [7]. With LC, as well as with HC, we avoid the frequently used empirical treatment of boundary value problems defined on the infinite domains based on the arbitrary truncation of the domain.

In order to march in time we have used gradually one step explicit methods such as Runge–Kutta (RK) methods of orders 2, 3 and 4, and linear multistep methods such as Adams–Bashforth (AB for short), Adams–Moulton (AM for short) and leap frog schemes. The regions of stability of all these finite differences formulas along with the spectral properties of Hermite and sinc differentiation matrices of moderate orders N, i.e., $64 \leq N \leq 128$ have assured the numerical stability of MoL. The necessary and sufficient conditions of Lax-stability formulated in [9] and [16] have been rigorously observed.

A particular attention is paid to the conservation of energy integral over time. The time marching schemes as well as the spatial collocation schemes are evaluated with respect to this aspect. More precisely, the *scaling parameter* involved in the HC has been adjusted in order to optimize the conservation of this invariant. Meantime this scaling factor has also been chosen in