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Higher-order efficient class of Chebyshev–Halley type methods

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ABSTRACT

Construction of two-point sixth-order methods for simple root is an ambitious and challenging task in numerical analysis. Therefore, the main aim of this paper is to introduce a new highly efficient two-point sixth-order class of Chebyshev–Halley type methods free from second-order derivative for the first time. Each member of the proposed class requires only four functional evaluations (viz. two evaluations of function f and two of first-order derivative f') per full iteration. A variety of concrete numerical examples illustrate that our proposed methods are more efficient and perform better than existing two-point/three-point sixth-order methods available in the literature. From their dynamical study, it has been observed that our proposed methods have better stability and robustness as compared to the other existing methods.

1. Introduction

Efficient solution techniques are required for finding simple roots of nonlinear equation of the form

$$f(x) = 0, \tag{1.1}$$

which partake of scientific, engineering and various other models. One of the best known one-point optimal method is the classical Newton method [1]. Many methods have been developed which improve the convergence rate of the Newton method at the expense of additional evaluations of functions, derivatives and changes in the points of iterations. All these modifications are in the direction of increasing the local order of convergence with the view of increasing their efficiency indices.

In 2000, Hernández [2] proposed a family of methods which is an improvement over Newton's method, defined as below:

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\,\frac{L_f(x_n)}{1 - \alpha L_f(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}, \tag{1.2}$$

where $L_f(x_n) = \frac{f''(x_n) f(x_n)}{f'^2(x_n)}$.

This family is known to be cubically convergent and some famous iterative methods can be reported as its particular cases. For example, the classical Chebyshev method [1,3] can be obtained if $\alpha = 0$, Halley's method [1,3] can be obtained if $\alpha = \frac{1}{2}$ and super-Halley method [1,3] can be obtained if $\alpha = 1$, respectively.

With the advancements of computer algebra, a number of sixth-order methods are also appearing as the extensions of Newton's method or Newton like method. In [4], Sharma and Guha proposed three-point family of sixth-order methods based on fourth-order Ostrowski's method [5] as follows:

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$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n)}{f(x_n) - 2f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(x_n) + b_1 f(y_n)}{f(x_n) + (b_1 - 2)f(y_n)} \dfrac{f(z_n)}{f'(x_n)}, \quad b_1 \in \mathbb{R}. \end{cases} \tag{1.3}$$

Recently, Chun [6] gave a three-point sixth-order family of Jarratt's method [7], for obtaining simple roots of nonlinear equations, which is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - J_f(x_n)\dfrac{f(x_n)}{f'(x_n)}, \quad J_f(x_n) = \dfrac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{b_2(z_n - y_n)(z_n - y_n) + \frac{3}{2}J_f(x_n)f'(y_n) + (1 - \frac{3}{2}J_f(x_n))f'(x_n)}, \quad b_2 \in \mathbb{R}. \end{cases} \tag{1.4}$$

On the other hand, Wang [8] has constructed a more general iteration scheme for simple roots, requiring four functional evaluations per iteration, which is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{9 - 5w}{10 - 6w}\dfrac{f(x_n)}{f'(y_n)}, \quad w = \dfrac{f'(y_n)}{f'(x_n)} \\ x_{n+1} = z_n - \dfrac{b_3 + b_4 w}{b_5 + b_6 w + b_7 w^2}\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{1.5}$$

where $b_3 = \frac{5b_5 + 3b_6 + b_7}{2}, \ b_4 = \frac{b_7 - 3b_5 - b_6}{2}, \ b_5 + b_6 + b_7 \neq 0, \ b_3, b_4, b_5, b_6, b_7 \in \mathbb{R}$.

But, the body structures of above mentioned three-point sixth-order methods requiring four functional evaluations are more complicated as compared to two-point methods [5,7]. Further, it is very rare to find two-point methods whose order of convergence is higher than four. Recently, Geum et al. [9] have developed a class of two-point sixth-order schemes in a fairly general form of weight functions with complex dynamics behind the basins of attraction. One particular member of this class is shown below:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n + \left(\dfrac{2s}{1 - 4s + s^2} + k^2\right)\dfrac{f(y_n)}{f'(y_n)}, \quad s = \dfrac{f'(y_n)}{f'(x_n)}, \quad k = \dfrac{f(y_n)}{f(x_n)}. \end{cases} \tag{1.6}$$

Obtaining new two-point methods of order six not requiring the computation of a second-order derivative is a very important and interesting task from the practical point of view. Although the schemes developed by Geum et al. have a general form of weight functions, in this manuscript we are interested in designing a new two-point sixth-order class of iterative methods from a view point of Chebyshev–Halley type methods, which do not require any second-order derivative evaluation for obtaining simple roots of nonlinear equations, numerically. Each family requires two evaluations of the given function and one evaluation of the derivative per iteration. It is also observed that the body structures of our proposed families of methods are simpler than the existing three-point families of sixth-order methods. Further, these families of iterative methods are more competent in all the tested examples to the existing three-point sixth-order methods available in the literature and in addition, we also compare them with two-point sixth-order methods that is very recently proposed by Guem et al. [9]. The dynamic study of these methods also supports the theoretical aspects.

2. Development of two-point sixth-order methods

In this section, our aim is to develop many new families of sixth-order Chebyshev–Halley type methods, not requiring the computation of second-order derivative. For this purpose, we consider the well known second-order Newton's method, which is defined as follows:

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.1}$$