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Generating functions and existence of contact symmetries of third order scalar ordinary differential equations

R. Campoamor-Stursberg*

IMI and Dpto de Geometría y Topología, Fac. CC. Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, E-28040 Madrid, Spain

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ABSTRACT

Analyzing the structure of generating functions associated to the contact symmetry condition of third order scalar ordinary differential equations, constraints guaranteeing that the ODE does not admit contact symmetries are obtained. Further, inspecting the subalgebras generated by contact symmetries having a specific form, an alternative procedure to generate third order equations with irreducible contact symmetry algebra $\mathfrak{sp}(4)$ is developed.

1. Introduction

Lie point symmetries of scalar ordinary differential equations (ODEs) have been studied in detail by many authors, and their structure and classification types are nowadays well known ([1–8] and references therein). For systems of ODEs, various general results also exist, although in this case the analysis is still far from being complete [9–13]. In this context, the notion of symmetries of differential equations more general than point symmetries, such as contact symmetries, was already considered by S. Lie in his pioneering work [1]. Various applications to Differential Geometry and physics, as well as the phenomenon of hidden symmetries, pointed out the necessity of studying and understanding in detail these generalizations of point symmetries [14–19]. Lie himself noted that, for many equations, contact symmetries could be reduced, via a local transformation, to a point symmetry on the transformed reference [20], introducing the notion of irreducible contact symmetries for the case where such a transformation is not possible. Lie also obtained a complete classification of irreducible contact algebras in two complex variables [20]. In the last years, various results on the classification of these symmetries and their use in reduction order problems have been obtained, showing that contact symmetries constitute an important tool in the Lie group analysis of differential equations. Although contact symmetries are infinite in number for scalar second order scalar equations, they still are of considerable importance. It was shown in [14] that contact symmetries are related with functionally independent first integrals of the equation, hence allowing a complete symmetry classification. From third order onwards, the dimension of the contact symmetry algebra is finite, and can be used to reduce the order of the equation and to determine the structure and properties of hidden symmetries, notably of types I and II [15,16]. Contact symmetry algebras of third and higher order ODEs were analyzed systematically in [21], leading to a characterization of equations admitting an irreducible contact symmetry algebra. In particular, it was shown that for third order equations the existence of irreducible contact symmetries is equivalent to the possibility of reducing the equation

* Tel.: +34 913944571.
E-mail address: rutwig@ucm.es

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to the free equation $\dddot{x}=0$. As a consequence, only ODEs having a symmetry algebra isomorphic to the symplectic algebra $\mathfrak{sp}(4)$ possess irreducible contact symmetries.

The main objective of this work is to study some general features concerning contact symmetries of scalar third order ordinary differential equations over the field of real numbers. Instead of computing the complete contact symmetry algebra for different types of equations, we focus on the structure of the generating function arising from the symmetry condition and inspect some of its properties, specifically under which conditions the structure of a (fixed) contact symmetry guarantees that an ODE only has a reducible contact symmetry algebra. In this frame, we consider the construction of third order non-linear equations that are invariant with respect to some fixed (not necessarily irreducible) contact symmetry. This approach allows to successively impose additional contact symmetries to non-linear equations, in order to analyze the Lie subalgebra generated by these symmetries. Basing on the commutation relations of these symmetries, we develop an alternative method to those already known that allows to construct non-linear third order ODEs possessing a maximal symmetry algebra isomorphic to $\mathfrak{sp}(4)$. This in particular shows that the shape of the generating function already contains information concerning the irreducibility of the contact symmetry algebra of a differential equation.

1.1. Contact symmetries of ODEs

To determine the symmetries of an ODE, we proceed using the vector field formulation (see e.g. [2,22]). Recall that a generic nth-order ODE $x^{(n)}=\omega(t,\dot{x},\ddot{x},\ldots,x^{(n-1)})$ can be rewritten in terms of a linear operator $\mathbf{A}$ as

$$\mathbf{A}f=\left(\frac{\partial}{\partial t}+\dot{x}\frac{\partial}{\partial x}+\ddot{x}\frac{\partial}{\partial \dot{x}}+\cdots+\omega\frac{\partial}{\partial x^{(n-1)}}\right)f=0. \tag{1}$$

A vector field $X=\xi(t,x)\frac{\partial}{\partial t}+\eta(t,x)\frac{\partial}{\partial x}$ is a point symmetry generator whenever its nth-prolongation $\dot{X}$ satisfies the commutator $[\dot{X},\mathbf{A}]=-\frac{d\xi}{dt}\mathbf{A}$. The Ansatz to find contact symmetries is formally very similar. Consider a vector field

$$X=\xi(t,x,\dot{x})\frac{\partial}{\partial t}+\eta(t,x,\dot{x})\frac{\partial}{\partial x}+\eta^{(1)}(t,x,\dot{x})\frac{\partial}{\partial \dot{x}} \tag{2}$$

and suppose that $\eta^{(i)}=\frac{d\eta^{(i-1)}}{dt}-\frac{d\xi}{dt}x^{(i)}$ holds for $1\leq i\leq n-1$. Then X will be called a contact symmetry generator of (1) if the prolongation $\dot{X}=X+\sum_{i=1}^{n-1}\eta^{(i)}(t,x,\dot{x},\ldots,x^{(i+1)})\frac{\partial}{\partial \dot{x}_i}$ fulfils the condition

$$[\dot{X},\mathbf{A}]=\left(-\frac{d\xi}{dt}\right)\mathbf{A}. \tag{3}$$

Since the function $\eta^{(1)}$ is assumed to be independent on $\ddot{x}$, the prolongation rule implies the constraint

$$\frac{\partial\eta}{\partial\dot{x}}-\dot{x}\frac{\partial\xi}{\partial\dot{x}}=0, \tag{4}$$

showing that ξ and η are not independent functions. It is thus convenient to introduce a generating function $\Omega(t,x,\dot{x})$ such that

$$\xi(t,x,\dot{x})=\frac{\partial\Omega}{\partial\dot{x}},\quad \eta(t,x,\dot{x})=\dot{x}\frac{\partial\Omega}{\partial\dot{x}}-\Omega,\quad \eta^{(1)}(t,x,\dot{x})=-\frac{\partial\Omega}{\partial t}-\dot{x}\frac{\partial\Omega}{\partial x}. \tag{5}$$

Hence all successive prolongations are described in terms of higher order derivatives of Ω [22]. According to the terminology used in [16], we shall call a contact symmetry intrinsic whenever $\frac{\partial\xi}{\partial\dot{x}}\neq 0$ or $\frac{\partial\eta}{\partial\dot{x}}\neq 0$ holds. Contact symmetries (intrinsic and point) generate a Lie algebra $\mathcal{L}$ that we call the contact symmetry algebra of the equation. However, the intrinsic contact symmetries generally do not generate a subalgebra of $\mathcal{L}$. We also observe that an intrinsic contact symmetry is not necessarily irreducible [21]. A reducible (intrinsic) contact symmetry indicates that the ODE has not been presented in the most suitable system of coordinates. However, as the choice of coordinates is often conditioned to physical requirements, reducible intrinsic contact symmetries are still of interest for the problem under scrutiny.

Using the generating function $\Omega(t,x,\dot{x})$, the contact symmetry condition for third order ODEs $\dddot{x}=\omega(t,x,\dot{x},\ddot{x})$ is explicitly given by the partial differential equation

$$\begin{aligned}&-\frac{\partial\omega}{\partial\dot{x}}\frac{\partial\Omega}{\partial t}+\left(\omega-\dot{x}\frac{\partial\omega}{\partial\dot{x}}-\ddot{x}\frac{\partial\omega}{\partial\ddot{x}}\right)\frac{\partial\Omega}{\partial x}+\left(\frac{\partial\omega}{\partial t}+\dot{x}\frac{\partial\omega}{\partial x}\right)\frac{\partial\Omega}{\partial\dot{x}}-\frac{\partial\omega}{\partial\ddot{x}}\frac{\partial^2\Omega}{\partial t^2}+\left(3\ddot{x}-2\dot{x}\frac{\partial\omega}{\partial\ddot{x}}\right)\frac{\partial^2\Omega}{\partial t\partial x}\\&+\left(3\omega-2\ddot{x}\frac{\partial\omega}{\partial\ddot{x}}\right)\frac{\partial^2\Omega}{\partial t\partial\dot{x}}+\left(3\dot{x}\ddot{x}-\dot{x}^2\frac{\partial\omega}{\partial\ddot{x}}\right)\frac{\partial^2\Omega}{\partial x^2}+\left(3\dot{x}\omega+3\ddot{x}^2-2\dot{x}\ddot{x}\frac{\partial\omega}{\partial\ddot{x}}\right)\frac{\partial^2\Omega}{\partial x\partial\dot{x}}+\frac{\partial^3\Omega}{\partial t^3}\\&+\left(3\ddot{x}\omega-\ddot{x}^2\frac{\partial\omega}{\partial\ddot{x}}\right)\frac{\partial^2\Omega}{\partial\dot{x}^2}+3\dot{x}\frac{\partial^3\Omega}{\partial t^2\partial x}+3\ddot{x}\left(\frac{\partial^3\Omega}{\partial t^2\partial\dot{x}}+\ddot{x}\frac{\partial^3\Omega}{\partial t\partial\dot{x}^2}\right)+3\dot{x}^2\left(\frac{\partial^3\Omega}{\partial t\partial x^2}+\ddot{x}\frac{\partial^3\Omega}{\partial x^2\partial\dot{x}}\right)\\&+3\dot{x}\ddot{x}\left(\ddot{x}\frac{\partial^3\Omega}{\partial x\partial\dot{x}^2}+2\frac{\partial^3\Omega}{\partial t\partial x\partial\dot{x}}\right)+\dot{x}^3\frac{\partial^3\Omega}{\partial x^3}+\ddot{x}^3\frac{\partial^3\Omega}{\partial\dot{x}^3}-\frac{\partial\omega}{\partial x}\Omega=0.\end{aligned} \tag{6}$$