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Floquet theory based on new periodicity concept for hybrid systems involving *q*-difference equations[☆]

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Using the new periodicity concept based on shifts, we construct a unified Floquet theory for homogeneous and nonhomogeneous hybrid periodic systems on domains having continuous, discrete or hybrid structure. New periodicity concept based on shifts enables the construction of Floquet theory on hybrid domains that are not necessarily additive periodic. This makes periodicity and stability analysis of hybrid periodic systems possible on non-additive domains. In particular, this new approach can be useful to know more about Floquet theory for linear q-difference systems defined on $\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. By constructing the solution of matrix exponential equation we establish a canonical Floquet decomposition theorem. Determining the relation between Floquet multipliers and Floquet exponents, we give a spectral mapping theorem on closed subsets of reals that are periodic in shifts. Finally, we show how the constructed theory can be utilized for the stability analysis of dynamic systems on periodic time scales in shifts.

1. Introduction

The theory of periodic systems has taken a prominent attention in the existing literature due to its tremendous application potential in engineering, biology, biomathematics, chemistry etc. Floquet theory is an important tool for the investigation of periodic solutions and stability analysis of dynamic systems. Floquet theory of differential and difference systems can be found in [22,23], respectively. Floquet theory of Volterra equation has been handled in [10]. An extension of the Floquet theory to the systems with memory has been studied in [11]. In [7], Floquet theory has been employed for stability analysis of nonlinear integro-differential equations. Moreover, a generalization of Floquet theory in continuous case is studied in [28].

Providing a wide perspective to discrete and continuous analysis, time scale calculus is a useful theory for the unification of differential and difference systems. For the sake of brevity, we suppose familiarity with fundamental theory of time scales. For a comprehensive review on time scale theory, we may refer readers to [12,13]. Unification of discrete and continuous dynamic systems under the theory of time scales avoids the separate studies for differential and difference systems by using the similar arguments. Motivated by unification and extension capabilities of time scale calculus, the researchers in recent years have been developing the time scale analogues of existing results for difference, q-difference, and differential equations. For instance in [9], the authors construct a Floquet theory for additive periodic time scales and focus on Putzer representations of matrix logarithms. We use the terminology "additive periodic time scale" to refer to an arbitrary, closed, non-empty subset $\mathbb{T}$ of reals satisfying the

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following property ([21]):

$$\text{there exists a fixed } P \in \mathbb{T} \text{ such that } t \pm P \in \mathbb{T} \text{ for all } t \in \mathbb{T}. \tag{1.1}$$

In [17], DaCunha unified Floquet theory for nonautonomous linear dynamic systems based on Lyapunov transformations by using matrix exponential on time scales (see [12, Section 5]). Afterward, DaCunha and Davis improved the results of [17] in [16]. Note that the results in [16,17] regarding Floquet theory are valid only on additive periodic time scales. This strong restriction prevents investigation of periodicity on very important particular time scales. For instance, the q-difference equations are established on the time scale

$$\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\}, \quad q > 1$$

which is not additive periodic. Hence, the existing unified Floquet theory does not cover the systems of q-difference equations. A q-difference equation is an equation including a q-derivative D_q, given by

$$D_q(f)(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \in q^{\mathbb{Z}},$$

of its unknown function. Observe that the q-derivative $D_q(f)$ of a function f turns into ordinary derivative f' if we let $q \to 1$. The theory of q-difference equations is a useful tool for the discretization of differential equations used for modeling continuous processes (see e.g. [19,24,25], and references therein). In [27] the author says "*in the p-adic context, q-difference equations are not simply a discretization of solutions of differential equations, but they are actually equal*". We may also refer to [8] for further discussion about the equivalence between q-difference equations and differential equations. There is a vast literature on the existence of periodic solutions of differential equations, unlike the existence of periodic solutions of q-difference equations. Thus, it is of importance to study the existence of periodic solutions of q-difference equations.

In recent years, the shift operators, denoted $\delta_{\pm}(s, t)$, are introduced to construct delay dynamic equations and a new periodicity concept on time scales (see [1,4,5]). We give a detailed information about the shift operators in further sections. We may also refer to the studies [2,3,5] for the basic definitions, properties and some applications of shift operators on time scales. In particular, we direct the readers to [1] for the construction of new periodicity concept on time scales. The motivation of new periodicity concept in [1] stems from the following ideas:

- I.1. Addition is not always the only way to step forward and backward on a time scale, for instance, the operators $\delta_{\pm}(2, t) = 2^{\pm 1}t$ determine backward and forward shifts on the time scale $\{2^n : n \in \mathbb{Z}\} \cup \{0\}$.
- I.2. We may use shift operators $\delta_{\pm}$ with certain properties to obtain a backward and forward motion on a general time scale. Similar to (1.1) a periodic time scale in shifts can be defined to be the one satisfying the following property:

$$\text{there exists a fixed } P \in \mathbb{T} \text{ such that } \delta_{\pm}(P, t) \in \mathbb{T} \text{ for all } t \in \mathbb{T}. \tag{1.2}$$

This approach enables the study of periodicity notion on a large class of time scales that are not necessarily additive periodic. For instance, the time scale $\overline{q^{\mathbb{Z}}}$ is periodic in shifts $\delta_{\pm}(s, t) = s^{\pm}t$ since

$$\delta_{\pm}(q, t) = q^{\pm}t \in \mathbb{T} \text{ for all } t \in \mathbb{T}.$$

Therefore, one may define a q^k-periodic function f on $\overline{q^{\mathbb{Z}}}$ as follows:

$$f\left(q^{\pm k}t\right) = f(t) \text{ for all } t \in \overline{q^{\mathbb{Z}}} \text{ and a fixed } k \in \{1, 2, \ldots\}.$$

More generally, a T-periodic function f on a P-periodic time scale $\mathbb{T}$ in shifts $\delta_{\pm}$ can be defined as follows

$$f(\delta_{\pm}(T, t)) = f(t) \text{ for all } t \in \mathbb{T} \text{ and a fixed } T \in [P, \infty) \cap \mathbb{T}.$$

In this paper, we use Lyapunov transformation (see [16, Definition 2.1]) and the new periodicity concept developed in [1] to construct a unified Floquet theory for hybrid systems on hybrid domains. As an alternative to the existing literature, our Floquet theory and stability results are valid on more time scales, such as $\overline{q^{\mathbb{Z}}}$ and

$$\cup_{k=1}^{\infty} [3^{\pm k}, 2.3^{\pm k}] \cup \{0\}$$

which cannot be covered by [16,17]. It should be mentioned that periodicity notion and Floquet theory on the time scale

$$q^{\mathbb{N}_0} = \{q^n : q > 1 \text{ and } n = 0, 1, 2, \ldots\}$$

have been studied in [14,15]. In [14,15] a ω-periodic function f on $q^{\mathbb{N}_0}$ is defined to be the one satisfying

$$f(q^{\omega}t) = \frac{1}{q^{\omega}} f(t) \text{ for all } t \in q^{\mathbb{N}_0} \text{ and a fixed } \omega \in \{1, 2, \ldots\}.$$

According to this periodicity definition the function $g(t) = 1/t$ is q-periodic over the time scale $q^{\mathbb{N}_0}$. Unlike the conventional periodic functions in the existing literature, the function $g(t) = 1/t$ does not repeat its values at each period t, $q^{\omega}t$, $(q^{\omega})^2 t$,.... In parallel with conventional periodicity perception, we define a periodic function to be the one repeating its values at each forward/backward step on its domain with a certain size. For instance, according to our definition the function $h(t) = (-1)^{\frac{\ln t}{\ln q}}$ is a q^2-periodic function on $q^{\mathbb{Z}} = \{q > 1 : q^n, n \in \mathbb{Z}\}$ since

$$h(\delta_{\pm}(q^2, t)) = (-1)^{\frac{\ln t}{\ln q} \pm 2} = (-1)^{\frac{\ln t}{\ln q}} = h(t).$$