Applied Mathematics and Computation 272 (2016) 20–27

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Limits of level and parameter dependent subdivision schemes: A matrix approach

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ARTICLE INFO

Article history:
Available online 1 October 2015

Keywords:
Level dependent (non-stationary) subdivision schemes
Tension parameter
Sum rules
Hölder regularity
Joint spectral radius

ABSTRACT

In this paper, we present a new matrix approach for the analysis of subdivision schemes whose non-stationarity is due to linear dependency on parameters whose values vary in a compact set. Indeed, we show how to check the convergence in $C^\ell(\mathbb{R}^s)$ and determine the Hölder regularity of such level and parameter dependent schemes efficiently via the joint spectral radius approach. The efficiency of this method and the important role of the parameter dependency are demonstrated on several examples of subdivision schemes whose properties improve the properties of the corresponding stationary schemes.

1. Introduction

We analyze convergence and Hölder regularity of multivariate level dependent (non-stationary) subdivision schemes whose masks depend linearly on one or several parameters. For this type of schemes, which include well-known schemes with tension parameters [1,12,14,23,24,32], the theoretical results from [7] are applicable, but not always efficient. Indeed, if the level dependent parameters vary in some compact set, then the set of the so-called limit points (see [7]) of the corresponding sequence of non-stationary masks exists, but cannot be determined explicitly. This hinders the regularity analysis of such schemes. Thus, we present a different perspective on the results in [7] and derive a new general method for convergence and regularity analysis of such level and parameter dependent schemes. The practical efficiency of this new method is illustrated in several examples.

Subdivision schemes are iterative algorithms for generating curves and surfaces from given control points of a mesh. They are easy to implement and intuitive in use. These and other nice mathematical properties of subdivision schemes motivate their popularity in applications, i.e. in modeling of freeform curves and surfaces, approximation and interpolation of functions, computer animation, signal and image processing etc. Non-stationary subdivision schemes extend the variety of different shapes generated by stationary subdivision. Indeed, the level dependency enables to generate new classes of functions such as exponential polynomials, exponential B-splines, etc. This gives a new impulse to development of subdivision schemes and enlarges the scope of their applications, e.g. in biological imaging [17,36], geometric design [34,37] or isogeometric analysis [2,9].

The main challenges in the analysis of any subdivision scheme are its convergence (in various function spaces), the regularity of its limit functions and its generation and reproduction properties. The important role of the matrix approach for regularity

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http://dx.doi.org/10.1016/j.amc.2015.08.120
0096-3003/

analysis of stationary subdivision schemes is well-known. It allows to reduces the analysis to the computation or to the estimation of the joint spectral radius of the finite set of square matrices derived from the subdivision mask. Recent advances in the joint spectral radius computation [26,33] make the matrix approach very precise and efficient. In the non-stationary setting, however, this approach has never been applied because of the several natural obstacles. First of all, the matrix products that emerge in the study of non-stationary schemes have a different form than those usually analyzed by the joint spectral radius techniques. Secondly, the masks of non-stationary schemes do not necessarily satisfy sum rules, which destroys the relation between the convergence of the scheme and spectral properties of its transition matrices. All those difficulties were put aside by the results in [7], where the matrix approach was extended to the non-stationary setting.

In this paper, in Section 3, we make the next step and consider level and parameter dependent subdivision schemes whose masks include tension parameters, used to control the properties of the subdivision limit. Mostly, the tension parameters are level dependent and influence the asymptotic behavior of the scheme. If this is the case, the scheme can be analyzed by Charina et al. [7], which states that the convergence and Hölder regularity of any such non-stationary scheme depends on the joint spectral radius of the matrices generated by the so-called limit points of the sequence of level-dependent masks. In Theorem 3.5, we show that for the schemes with linear dependence on these parameters, the result of [7] can be simplified and be made more practical, see examples in Section 3.1.

2. Background

Let $mI \in \mathbb{Z}^{s\times s}$, $|m| \geq 2$, be a dilation matrix and $E = \{0, \dots, |m|-1\}^s$ be the set of the coset representatives of $\mathbb{Z}^s/m\mathbb{Z}^s$. We study subdivision schemes given by the sequence $\{S_{\mathbf{a}^{(r)}},\ r \in \mathbb{N}\}$ of subdivision operators $S_{\mathbf{a}^{(r)}} : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ that define the subdivision rules by

$$(S^{(r)}_{\mathbf{a}}\mathbf{c})_\alpha = \sum_{\beta\in\mathbb{Z}^s} \mathrm{a}^{(r)}_{\alpha-m\beta} c_\beta, \quad \alpha \in \mathbb{Z}^s.$$

The masks $\mathbf{a}^{(r)} = \{\mathrm{a}^{(r)}_\alpha \in \mathbb{R},\ \alpha \in \mathbb{Z}^s\}$, $r \in \mathbb{N}$, are sequences of real numbers and are assumed to be all supported in $\{0, \dots, N\}^s$, $N \in \mathbb{N}$. For the set

$$K = \sum_{r=1}^{\infty} m^{-r} G, \quad G = \{-|m|, \dots, N+1\}^s, \tag{2.1}$$

the masks define the square matrices

$$A^{(r)}_{\varepsilon,\mathrm{a}^{(r)}} = \left(\mathrm{a}^{(r)}_{m\alpha+\varepsilon-\beta}\right)_{\alpha,\beta\in K}, \quad r \in \mathbb{N}, \quad \varepsilon \in E. \tag{2.2}$$

We assume that the level dependent symbols

$$a^{(r)}(z) = \sum_{\alpha\in\mathbb{Z}^s} \mathrm{a}^{(r)}_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1}\cdot \ldots \cdot z_s^{\alpha_s}, \quad z \in (\mathbb{C}\setminus\{0\})^s,$$

of the subdivision scheme

$$c^{(r+1)} = S_{\mathbf{a}^{(r)}} c^{(r)} = S_{\mathbf{a}^{(r)}} S_{\mathbf{a}^{(r-1)}} \dots S_{\mathbf{a}^{(1)}} c^{(1)}, \quad r \in \mathbb{N}, \quad c^{(1)} \in \ell(\mathbb{Z}^s),$$

satisfy sum rules of order $\ell+1$, $\ell \in \mathbb{N}_0$.

Definition 2.1. Let $\ell \in \mathbb{N}_0$, $r \in \mathbb{N}$. The symbol $a^{(r)}(z), z \in (\mathbb{C}\setminus\{0\})^s$, satisfies sum rules of order $\ell+1$ if

$$a^{(r)}(1, \dots, 1) = |m|^s \quad \text{and} \quad \max_{|\eta|\leq \ell} \max_{\epsilon\in\Xi\setminus\{1\}} |D^\eta a^{(r)}(\epsilon)| = 0\,, \tag{2.3}$$

where $\Xi = \{e^{-i\frac{2\pi}{|m|}\varepsilon} = (e^{-i\frac{2\pi}{|m|}\varepsilon_1}, \dots, e^{-i\frac{2\pi}{|m|}\varepsilon_s}),\ \varepsilon \in E\}$ and $D^\eta = \frac{\partial^{\eta_1}}{\partial z_1^{\eta_1}} \cdots \frac{\partial^{\eta_s}}{\partial z_s^{\eta_s}}$.

For more details on sum rules see e.g [3,4,30].

Remark 2.2. (i) The assumption that all symbols $a^{(r)}(z)$ satisfy sum rules of order $\ell+1$, guarantees that the matrices $A^{(r)}_{\varepsilon,\mathrm{a}^{(r)}}$, $\varepsilon \in E$, $r \in \mathbb{N}$, in (2.2) have common left-eigenvectors of the form

$$(p(\alpha))_{\alpha\in K}, \quad p \in \Pi_\ell,$$

where Π_ℓ is the space of polynomials of degree less than or equal to ℓ. Thus, the matrices $A^{(r)}_{\varepsilon,\mathrm{a}^{(r)}}$, $\varepsilon \in E, r \in \mathbb{N}$, possess a common linear subspace $V_\ell \subset \mathbb{R}^{|K|}$ orthogonal to the span of the common left-eigenvectors of $A^{(r)}_{\varepsilon,\mathrm{a}^{(r)}}$, $\varepsilon \in E, r \in \mathbb{N}$. The spectral properties of the set

$$\mathcal{T} = \{A^{(r)}_{\varepsilon,\mathrm{a}^{(r)}}|_{V_\ell},\ \varepsilon \in E,\ r \in \mathbb{N}\}$$

determine the regularity of the non-stationary scheme, see [7].