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A family of smooth and interpolatory basis functions for parametric curve and surface representation☆

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ABSTRACT

Interpolatory basis functions are helpful to specify parametric curves or surfaces that can be modified by simple user-interaction. Their main advantage is a characterization of the object by a set of control points that lie on the shape itself (*i.e.*, curve or surface). In this paper, we characterize a new family of compactly supported piecewise-exponential basis functions that are smooth and satisfy the interpolation property. They can be seen as a generalization and extension of the Keys interpolation kernel using cardinal exponential B-splines. The proposed interpolators can be designed to reproduce trigonometric, hyperbolic, and polynomial functions or combinations of them. We illustrate the construction and give concrete examples on how to use such functions to construct parametric curves and surfaces.

1. Introduction

The representation of shapes using parametric curves and surfaces is widely used in domains that make use of computer graphics [1,2] such as industrial design [3–5], the animation industry [6], as well as for the analysis of biomedical images [7–9]. In that context, it is often important to be able to interactively change the shape of the curve or surface. The spline-based representation of parametric shapes has proven to be a convenient choice to include user interactivity in shape modeling due to the underlying control-point-based nature of spline functions. If the basis functions are compactly supported, the change of position of a control point modifies the shape only locally. This allows for a local control by the user. Commonly used basis functions such as NURBS or B-splines have this locality property but are in general not interpolatory (except for example zeroth and first degree B-splines, which are not smooth) [10]. This has the disadvantage that the control points do not directly lie on the contour or surface of the shape. Especially in 3D applications, this can be inconvenient because it is no longer intuitive to interactively modify complex shapes. More recently a method to construct piecewise polynomial interpolators has been presented in [11,12].

In this paper, we propose a new family of piecewise exponential basis functions that are interpolatory and are at least in $\mathcal{C}^1$. They are compactly supported and their order can be chosen to be arbitrarily high. We show that they are able to reproduce exponential polynomials which include the pure polynomials as a subset. This convenient property is particularly relevant for the exact rendering of conic sections such as circles, ellipses, or parabolas, as well as other trigonometric and hyperbolic curves and surfaces [13]. In its absence, one must resort to subdivision to tackle this aspect [14–18]. However, existing comparable subdivision schemes usually rely on basis functions that are defined as a limit process and do not have a closed-form expression [19].

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Our proposed family generalizes the piecewise-polynomial Keys interpolator [20–22] to higher degrees and can be seen as its extension using exponential B-splines [23,24].

The paper is organized as follows. In Section 2 we give a brief review on exponential B-splines and their relation with differential operators. This is needed to understand the reproduction properties of our proposed interpolators since they are based on exponential B-splines. In Section 3 we present the proposed family of interpolators. We present the relevant properties and prove that they reproduce exponential polynomials. We also provide a generic algorithm to construct specific interpolators that belong to the proposed family. In Section 4 we give specific examples of interpolators and we explicitly show how parametric curves and surfaces with desirable reproduction properties are constructed.

2. Exponential B-splines

In this section, we briefly review the link between exponential B-splines and differential operators which is crucial to understand the reproduction properties of the proposed spline family. (For a more in-depth characterization of exponential B-splines, we refer the reader to [23,24].) These reproduction properties are needed for the exact representation of elementary shapes (see Section 4.3 for examples) and are automatically enforced by our construction.

2.1. Notations

We describe the list of roots $\alpha_1, \dots, \alpha_N$ using the vector notation $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_N)$. To assert the inclusion of a list of roots $\boldsymbol{\alpha}_1$ into another list $\boldsymbol{\alpha}_2$, we use the set notation $\boldsymbol{\alpha}_1 \subset \boldsymbol{\alpha}_2$. If $\boldsymbol{\alpha}_1$ must be excluded from $\boldsymbol{\alpha}_2$, we write $\boldsymbol{\alpha}_2 \backslash \boldsymbol{\alpha}_1$. Similarly, we denote the union of the two lists of roots $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ by $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$. Likewise, we write $\alpha_n \in \boldsymbol{\alpha}$ to signify that one of the components of $\boldsymbol{\alpha}$ is α_n. Furthermore, the nth-order derivative operator is denoted by $D^n = \frac{\mathrm{d}^n}{\mathrm{d}t^n}$ with $D^0 = I$ (identity operator).

2.2. Operator properties and reproduction of null-space components

Consider the generic differential operator L of order N

$$L = D^N + a_{N-1} D^{N-1} + \dots + a_0 I. \tag{1}$$

Its characteristic polynomial with variable $s \in \mathbb{C}$ is given by

$$L(s) = s^N + a_{N-1} s^{N-1} + \dots + a_0 = \prod_{n=1}^{N} (s - \alpha_n). \tag{2}$$

By evaluating $L(s)$ at $s = \mathrm{j}\omega$, where $\mathrm{j}^2 = -1$, we obtain that the frequency response of the differential operator is $\hat{L}(\mathrm{j}\omega) = \prod_{n=1}^{N} (\mathrm{j}\omega - \alpha_n)$. This allows us to factorize the operator L as

$$L_{\boldsymbol{\alpha}} := L = (D - \alpha_1 I)(D - \alpha_2 I) \cdots (D - \alpha_N I). \tag{3}$$

It follows that the nullspace, which contains all the solutions of the homogenous differential equation $L_{\boldsymbol{\alpha}}\{f_0\}(t) = 0$, is given by

$$N_{L_{\boldsymbol{\alpha}}} = \mathrm{span}\{t^{n-1} \mathrm{e}^{\alpha_{(m)} t}\}_{m=1,\dots,N_d; n=1,\dots,n_{(m)}}, \tag{4}$$

where the N_d distinct roots of the characteristic polynomial are denoted by $\alpha_{(1)}, \dots, \alpha_{(N_d)}$ with the multiplicity of $\alpha_{(m)}$ being $n_{(m)}$ and $\sum_{m=1}^{N_d} n_{(m)} = N$. There exists a unique causal Green's function $\rho_{\boldsymbol{\alpha}}$ ($\rho_{\boldsymbol{\alpha}}(t) = 0$ for $t < 0$) associated to the operator $L_{\boldsymbol{\alpha}}$ that satisfies $L_{\boldsymbol{\alpha}}\{\rho_{\boldsymbol{\alpha}}\}(t) = \delta(t)$, where δ is the Dirac distribution.

Its explicit form is

$$\rho_{\boldsymbol{\alpha}}(t) = \sum_{m=1}^{N_d} \sum_{n=1}^{n_{(m)}} c_{m,n} \frac{t_+^{n-1}}{(n-1)!} \mathrm{e}^{\alpha_{(m)} t}, \tag{5}$$

with suitable constants $c_{m,n}$. We see that (5) is a causal exponential polynomial. The discrete counterpart of $L_{\boldsymbol{\alpha}}$ is denoted by $\Delta_{\boldsymbol{\alpha}}$. It is specified by its symbol $\hat{\Delta}_{\boldsymbol{\alpha}}(z) = \prod_{n=1}^{N} (1 - \mathrm{e}^{\alpha_n} z^{-1})$. An exponential B-spline is then defined as $\beta_{\boldsymbol{\alpha}}(t) = \Delta_{\boldsymbol{\alpha}}\{\rho_{\boldsymbol{\alpha}}\}(t)$, which is equivalent to the Fourier-domain definition

$$\hat{\beta}_{\boldsymbol{\alpha}}(\omega) = \frac{\hat{\Delta}_{\boldsymbol{\alpha}}(\mathrm{e}^{\mathrm{j}\omega})}{\hat{L}_{\boldsymbol{\alpha}}(\mathrm{j}\omega)} = \prod_{k=1}^{n} \frac{1 - \mathrm{e}^{\alpha_k - \mathrm{j}\omega}}{\mathrm{j}\omega - \alpha_k}. \tag{6}$$

Since $\Delta_{\boldsymbol{\alpha}}$ is defined on the integer grid, the exponential B-splines reproduce the causal Green's function (5) associated to $L_{\boldsymbol{\alpha}}$

$$\rho_{\boldsymbol{\alpha}}(t) = \Delta_{\boldsymbol{\alpha}}^{-1}\{\beta_{\boldsymbol{\alpha}}\}(t) = \sum_{k=0}^{+\infty} p_{\boldsymbol{\alpha}}[k] \beta_{\boldsymbol{\alpha}}(t - k), \tag{7}$$

where $p_{\boldsymbol{\alpha}}$ is a unique causal sequence as has been shown in [23]. Extrapolating the Green's function (5) for $t < 0$ is equivalent to extrapolating the sum in (7) for negative k, which results in the reproduction of an exponential polynomial. More generally, it can be shown that $\beta_{\boldsymbol{\alpha}}$ is able to reproduce any component $P_0(t) \in N_L$ that is in the null space of $L = L_{\boldsymbol{\alpha}}$.