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A nonlinear algorithm for monotone piecewise bicubic interpolation☆

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ABSTRACT

In this paper we present an algorithm for monotone interpolation of monotone data on a rectangular mesh by piecewise bicubic functions. Carlton and Fritsch (1985) develop conditions on the Hermite derivatives that are sufficient for such a function to be monotone. Here we extend our results of Aràndiga (2013) to obtain nonlinear approximations to the first partial and first mixed partial derivatives at the mesh points that allow us to construct a monotone piecewise bicubic interpolants. We analyze its order of approximation and present some numerical experiments.

1. Introduction

Approximation methods used in some scientific and engineering problems (chemistry, robotics, etc.) are required to represent the physical reality as accurately as possible i.e. they should be capable of reproducing the existing data with high precision. In addition, very often it is recommended that the approximation techniques employed are also "shape preserving" i.e. the approximating function preserves certain properties of the data such as monotonicity or convexity.

In this paper we propose and analyze an algorithm which produces monotone piecewise bicubic Hermite interpolants. Since the interpolant is piecewise cubic, one expects that such algorithms provide a fourth order l_∞ approximation whenever the interpolated function is sufficiently smooth. However, if the algorithm is linear, then, as shown in [13], it is at best first order accurate. Consequently, if greater accuracy is desired, the algorithm must be nonlinear.

Several monotonicity-preserving interpolatory methods based on cubic polynomials can be found in the literature ([2,3,6,14,15,18–20,22] in 1D and [5,7–11] in 2D). These have as a common characteristic that the order of accuracy of the interpolation procedure is diminished at those regions where monotonicity constraints are enforced. This seems to be a common trend, as it is often necessary to lose accuracy in the interpolating function in order to ensure monotonicity, or, inversely, to lose monotonicity in order to ensure high order accuracy.

A way to construct piecewise bicubic interpolant to a given set of data consist in calculating the first partial and first mixed partial derivatives conditions at the mesh points and then constructing the bicubic Hermite interpolant. In [7–9] the authors gave necessary and sufficient on these derivatives such that the resulting bicubic polynomial is monotone.

In this paper, we propose a formula for the computation of first partial and twist derivatives at the mesh points that leads to Hermite-type reconstructions which are third order accurate and monotonicity preserving if certain conditions are satisfied. We have observed that in many examples we get a monotone interpolant (to monotone data). If this is not the case we propose

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a modification of the evaluation of the derivative values that seems to always lead to monotonicity preserving reconstructions, even though we cannot ensure that the sufficient conditions of [7–9] are satisfied. This modification leads, however, to a second order accurate reconstruction. We also present a version of the evaluation of the derivative values that satisfies the sufficient conditions of [7–9] and, hence, ensure monotonicity and that it is also second order accurate.

Related research is carried out in [5] where the authors obtain monotone reconstructions using quadratic splines defined over a suitable triangular partition that in some cases are second order accurate. In [10] and [11] a local scheme based on the idea of adaptive degrees for comonotone interpolation of data on a rectangular grid is proposed obtaining monotone interpolants which are second order accurate.

The work in this paper is divided into five sections. In Section 2 we review the classical bicubic Hermite interpolatory formula and provide criteria to ensure the preservation of monotonicity. Section 3 is devoted to designing a new method of computing derivative values and analyzing their monotonicity preservation properties. Finally in the last two sections we present some numerical experiments and some conclusions.

2. Monotonicity preserving bicubic interpolation

2.1. Introduction

Suppose we know the data $f_{i,j} = f(x_i, y_j)$ on a rectangular mesh $X \otimes Y$ where $X = \{x_i\}_{i=1}^N$ and $Y = \{y_j\}_{j=1}^M$.

For many applications the data are monotone in the following sense:

$$sgn(f_{i+1,j} - f_{i,j}) = s_1, \; i = 1, \dots N-1, \, j = 1, \dots, M.$$
$$sgn(f_{i,j+1} - f_{i,j}) = s_2, \; j = 1, \dots M-1, \, i = 1, \dots, N;$$

where s_1 and s_2 are -1 or $+1$.

Our goal is to find a function $q(x, y)$ that is smooth and as accurate as possible on $R = [x_1, x_N] \times [y_1, y_M]$, which interpolates the data:

$$q(x_i, y_j) = f_{i,j}, \quad i = 1, \dots N, \, j = 1, \dots, M;$$

and which is monotone; that is,

$$sgn(q(x^*, y) - q(x, y)) = s_1 \text{ if } x_1 \le x < x^* \le x_N \text{ for all } y \in [y_1, y_M]$$

and

$$sgn(q(x, y^*) - q(x, y)) = s_2 \text{ if } y_1 \le y < y^* \le y_M \text{ for all } x \in [x_1, x_N].$$

We define $h_i = (x_{i+1} - x_i)$, $k_j = (y_{j+1} - y_j)$ and $h = \max_{i,j}\{h_i, k_j\}$.

In classical Hermite interpolation the values of a function f, $f_{i,j} = f(x_i.y_j)$, and its derivatives, $\{\frac{\partial}{\partial x} f(x_i, y_j)\}$, $\{\frac{\partial}{\partial y} f(x_i, y_j)\}\{\frac{\partial^2}{\partial x \partial y} f(x_i, y_j)\}$, are interpolated at mesh points. Piecewise bicubic Hermite interpolants of smooth functions constructed in this manner are fourth order accurate (see [17] and references therein). That is,

$$q(x, y) = f(x, y) + \mathcal{O}(h^4).$$

If the derivatives are only approximated, that is: $f^x_{i,j} = \frac{\partial}{\partial x} f(x_i, y_j) + \mathcal{O}(h^{r_x})$, $f^y_{i,j} = \frac{\partial}{\partial y} f(x_i, y_j) + \mathcal{O}(h^{r_y})$ and $f^{xy}_{i,j} = \frac{\partial^2}{\partial x \partial y} f(x_i, y_j) + \mathcal{O}(h^{r_{xy}})$ then

$$q(x, y) = f(x, y) + \mathcal{O}(h^r) \quad \text{where } r = \min\{4, r_x + 1, r_y + 1, r_{xy} + 2\}.$$

The ideal case, then, is to have third order accurate approximations to these values, since these also lead to interpolants with maximum accuracy.

2.2. Interpolation formula

Given approximate values $\{f^x_{i,j}\}$, $\{f^y_{i,j}\}$ and $\{f^{xy}_{i,j}\}$ of $\{\frac{\partial}{\partial x} f(x, y)\}$, $\{\frac{\partial}{\partial y} f(x, y)\}\{\frac{\partial^2}{\partial x \partial y} f(x, y)\}$ at the nodes $\{(x_i, y_j)\}$, the piecewise cubic Hermite interpolant is (see [7])

$$q(x, y) = q_{i,j}(x, y) \quad \text{if} \quad (x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}], \tag{1}$$

where:

$$\begin{aligned} q_{i,j}(x, y) = \; & f_{i,j} q^i_{1,X}(x) q^j_{1,Y}(y) + f_{i+1,j} q^i_{2,X}(x) q^j_{1,Y}(y) \\ & + f_{i,j+1} q^i_{1,X}(x) q^j_{2,Y}(y) + f_{i+1,j+1} q^i_{2,X}(x) q^j_{2,Y}(y) \\ & + f^x_{i,j} q^i_{3,X}(x) q^j_{1,Y}(y) + f^x_{i+1,j} q^i_{4,X}(x) q^j_{1,Y}(y) \\ & + f^x_{i,j+1} q^i_{3,X}(x) q^j_{2,Y}(y) + f^x_{i+1,j+1} q^i_{4,X}(x) q^j_{2,Y}(y) \end{aligned}$$