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C^1 finite elements on non-tensor-product 2d and 3d manifolds

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ABSTRACT

Geometrically continuous (G^k) constructions naturally yield families of finite elements for isogeometric analysis (IGA) that are C^k also for non-tensor-product layout. This paper describes and analyzes one such concrete C^1 geometrically generalized IGA element (short: gIGA element) that generalizes bi-quadratic splines to quad meshes with irregularities. The new gIGA element is based on a recently-developed G^1 surface construction that recommends itself by its a B-spline-like control net, low (least) polynomial degree, good shape properties and reproduction of quadratics at irregular (extraordinary) points. Remarkably, for Poisson's equation on the disk using interior vertices of valence 3 and symmetric layout, we observe $O(h^3)$ convergence in the L^∞ norm for this family of elements. Numerical experiments confirm the elements to be effective for solving the trivariate Poisson equation on the solid cylinder, deformations thereof (a turbine blade), modeling and computing geodesics on smooth free-form surfaces via the heat equation, for solving the biharmonic equation on the disk and for Koiter-type thin-shell analysis.

1. Introduction

Isogeometric Analysis (IGA), as introduced in [13], is an isoparametric framework of numerical analysis that uses spline functions to represent both the geometric domain and the approximate solution of a partial differential equation (PDE). Where the partition of the geometric domain has *irregularities*, e.g. differs from the regular tensor-product spline lattice by having $n = 3$ or $n > 4$ quadrilateral domain pieces (patches) come together, the geometric design community has developed several extensions of the tensor-product (NURBS) representation. The two most widely-used representations are geometrically continuous (G^k) complexes of finitely many piecewise polynomial spline patches; and generalized subdivision surfaces that are defined by an infinite sequence of nested C^k surface rings and whose pieces can be viewed as *subdivision splines* with singularities at the irregular points [21]. Such subdivision splines have been used for finite element analysis as early as [8,9], but most recently received renewed attention, this time from the IGA community [1,18]. Since subdivision splines are naturally refinable, this approach is likely to gain a lot of traction once the computer-aided design community adopts subdivision into their design flow.

This paper focuses on the alternative approach of building isogeometric elements from geometrically continuous surface (G^k) constructions. This approach leverages the observation that any G^k construction yields C^k isogeometric finite elements by composing a G^k analysis function with the inverse of an equally G^k-parameterized physical domain [18,20] (see also the linear G^1 reparameterization in [16]). G^k complexes built from polynomial or rational tensor-product spline patches or patches in the

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Bernstein–Bézier form (BB-form) are automatically compatible with the industrial NURBS exchange standard, one of the goals of IGA [22,24].

In this paper, we specifically focus on a recently-developed G^1 construction [15] that extends bi-quadratic (bi-2) splines to more general quad meshes that can include non-4-valent points and multi-sided facets. The G^1 construction defines n patches of degree bi-3 in BB-form where $n = 3$ or $n = 5$ come together, respectively n patches of degree bi-4 where more than five patches join. (This distinction between the valences near the regular case of $n = 4$ and higher valences is both geometrically motivated and relevant in practice where the majority of irregularities are of valence 3 and 5.) The construction in [15] differs from the work in [4,22,25] in that it does not require solving equations while constructing the physical domain: the domain and the elements are modeled analogous to splines in B-spline form. That is, control points carry the geometric information and evaluation and differentiation amount to explicit formulas in terms of the control points. Shape optimization is integrated into the explicit formulas that relate the control net to their explicit piecewise Bézier representation. Rather than exploring the full space of available G^1 reparameterizations, as do two recent reports [4,14], we have selected a specific construction developed by careful consideration of the space of low-degree G-constructions.

Besides the low degree, our interest in developing this construction was piqued by the fact that, in order to create good reflection lines, the construction [15] offers high polynomial reproduction at the irregular points where n quadrilateral surface pieces meet. This property, known in the geometric design community as *flexibility*, is desirable both to increase smoothness and to ensure a rich gamut of shape. The observed numerical convergence shows that high flexibility is not only good for the surface quality but also for the numerical approximation order of gIGA elements: For a symmetric tessellation of the disk using interior vertices of valence 3 for Poisson's equation, we observe $O(h^3)$ convergence of the error not only in the L^2, but also in the L^∞ norm. This is in line with the optimal rate for bi-2 splines.

Overview. Section 2 reviews geometric continuity and Section 3 the derivation of the C^1 isogeometric element from a G^1 (surface) construction. Section 4 reports the performance of our implementation on several benchmark problems, including Poisson's equation (where smoothness of the elements is not needed), the heat equation on free-form surfaces (where smoothness is needed to model the surface), the biharmonic equation and thin place analysis (where the solution requires smooth elements).

2. Review of geometric continuity

Two C^k curve segments $\mathbf{x}_1 : [0..1] \to \mathbb{R}$ and $\mathbf{x}_2 : [0..1] \to \mathbb{R}$ join G^k at a common point $\mathbf{x}_1(1) = \mathbf{x}_2(0)$ if, possibly after a change of variables, derivatives match at the common point [5]. Generalizing this notion to edge-adjacent patches yields one of several equivalent notions of geometric continuity of surfaces as explained in the survey of geometric continuity [19, Section 3].

A convenient definition for the general multi-variable setup uses the classical notion of a k-jet of an $\mathbb{R}^d$-valued C^k map defined on an open neighborhood of a point $\mathtt{s} \in \mathbb{R}^m$, $m, d \geq 1$. This notion will help us to formally capture agreement of expansions of two maps at a common point or a set of common points forming a shared interface between two regions. For our application, $d, m \in \{2, 3\}$. For an integer $k \geq 1$, the k-jet is an equivalence class on the set of pairs

$$\mathcal{F}_{\mathtt{s},d} := \{(f, \mathcal{N}) \mid \mathcal{N} \text{ is an } \mathbb{R}^m\text{–open neighborhood of } \mathtt{s} \text{ and } f : \mathcal{N} \to \mathbb{R}^d \text{ is } C^k\}.$$

For each m-tuple $\mathbf{i} := (i_1, \ldots, i_m)$ consisting of non-negative integers i_j, define $|\mathbf{i}| := \Sigma i_j$ and let $\partial_\mathbf{i}$ denote the $|\mathbf{i}|$th-order partial-differentiation operator $(\frac{\partial}{\partial x_1})^{i_1}, \ldots, (\frac{\partial}{\partial x_m})^{i_m}$. The relation $\sim^k_\mathtt{s}$ on $\mathcal{F}_{\mathtt{s},d}$ defined by

$$(f_1, \mathcal{N}_1) \sim^k_\mathtt{s} (f_2, \mathcal{N}_2) \quad \text{if } \partial_\mathbf{i} f_1(\mathtt{s}) = \partial_\mathbf{i} f_2(\mathtt{s}) \text{ for all } \mathbf{i} \text{ with } |\mathbf{i}| \leq k,$$

is an equivalence relation, and the equivalence class of f under $\sim^k_\mathtt{s}$ is the *k-jet of f at* $\mathtt{s}$, denoted $\mathbf{j}^k_\mathtt{s} f$. Note that $|\mathbf{i}| = 0$ implies $f_1(\mathtt{s}) = f_2(\mathtt{s})$. Composition of jets is well-defined, also for jets on half-spaces that are used in our context of piecemeal-defined geometry.

The challenge addressed by geometric continuity is that the two maps whose jets are matched, each have their separate parameter domains. A change of variables, the reparameterization ρ, is needed to relate them to one another. More formally, for $\alpha = 1, 2$, let $\square_\alpha \subset \mathbb{R}^m$ be an m-dimensional polytope, for our applications a unit square or cube, and let E_α be an $(m-1)$-dimensional facet of $\square_\alpha$, with interior $\mathrm{int}(E_\alpha) = \overset{\circ}{E}_\alpha$ where $\overset{\circ}{P}$ denotes the interior with respect to the smallest space enclosing P. For our scenario, E_α is the interval $[0, \ldots, 1]$ when $m = 2$ and its tensor $[0..1]^2$ when $m = 3$. Then, following [12], we define $\rho : \mathcal{N}_1 \to \mathcal{N}_2$ to be a C^k diffeomorphism between two open sets $\mathcal{N}_1, \mathcal{N}_2 \subset \mathbb{R}^m$ that enclose $\overset{\circ}{E}_1$ and $\overset{\circ}{E}_2$ respectively such that $\rho(\overset{\circ}{E}_1) = \overset{\circ}{E}_2$, $\rho(\mathcal{N}_1 \cap \overset{\circ}{\square}_1) = \mathcal{N}_2 \setminus \square_2$ and $\rho(\mathcal{N}_1 \setminus \square_1) = \mathcal{N}_2 \cap \overset{\circ}{\square}_2$.

With the help of the reparameterization ρ, we can relate the two maps and define a G^k relation as follows. Let $\mathbf{x}_\alpha : \square_\alpha \subset \mathbb{R}^m \to \mathbb{R}^d$, $\alpha \in \{1, 2\}$ be C^k maps for which

$$\mathbf{x}_2(\rho(\mathtt{s})) = \mathbf{x}_1(\mathtt{s}) \quad \text{for all } \mathtt{s} \in \overset{\circ}{E}_1;$$

the images of the $\mathbf{x}_\alpha$ therefore join along a common interface $E := \mathbf{x}_2(E_2) = \mathbf{x}_1(E_1)$. We say that $\mathbf{x}_1$ *joins* $\mathbf{x}_2$ G^k *with reparameterization* ρ along E if for every $\mathtt{s} \in \overset{\circ}{E}_1$ we have

$$\mathbf{j}^k_\mathtt{s}\mathbf{x}_1 = \mathbf{j}^k_\mathtt{s}(\mathbf{x}_2 \circ \rho), \tag{1}$$