Applied Mathematics and Computation 272 (2016) 199–207

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Refinable functions, functionals, and iterated function systems

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ARTICLE INFO

Article history:
Available online 22 June 2015

MSC:
65D07
28A25
41A30
26A30

Keywords:
Refinable functions
Iterated function Systems
Invariant measures
Lattice equations

ABSTRACT

Invariant measures of iterated function systems and refinable functions are mathematical tools of vast usage, that, although usually considered in different contexts, are deeply linked. We outline these links, in particular with respect to existence and regularity of these objects.

1. Introduction

The importance in applications of real functions solving a refinement equation has led many authors to study their existence and properties. A partial list of these applications covers fractal interpolation, subdivision in Computer-Aided Geometric Design (CAGD) and attractors of iterated functions systems (IFSs), as considered for example in [2,12,25,45]. In many of these applications only the existence of an integral operator is required [6,14] so that L^1 regularity of refinable functions has been explored widely. Successively, in [33] the concept of refinability has been generalized to functionals. This is clearly related to previous works on the weak convergence of the subdivision process [21] and the existence of refinable distributions [14,17]. In these studies, the focus has shifted from refinable functions to refinable distributions.

In parallel, the study of IFSs [1,30] considers *invariant measures* satisfying a balance equation that is easily seen to be the dual of a functional refinement condition. Because of its original application to image processing, regularity of invariant measures is a theoretical problem that has received less attention. A remarkable exception to this state of things is provided by the study of Bernoulli convolutions [26,41,44], where the main theoretical question to be answered is about the absolute continuity versus singularity of the invariant measure. A renewed interest in this question has been brought recently by the study of IFS with uncountably many maps [36,37]: preliminary results on a specific example show that a transition from singular behavior to absolute continuity and to increasing regularity of the density of an IFS measure takes place as a parameter is varied. Crucial for our aim, this density is a refinable function.

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http://dx.doi.org/10.1016/j.amc.2015.06.006
0096-3003/

We can therefore appreciate that, whether going from refinable functions to functionals and distributions, or from IFS measures to their densities, we are in the presence of common problems, although expressed in different mathematical jargon. The scope of this paper is to outline as simply as possible the translation language between the two approaches, and to expose the main results that can be useful to carry from one side to the other, like *e.g* the techniques for the evaluation of the integrals [8,9,11,32,35]. Some previous work in this direction can be found in [34,40].

The plan of this paper is the following: In the next section we introduce the basic definitions of refinable functions and iterated function systems, and in Section 3 we show how to link one concept to the other. Formal manipulations are presented in Section 4, that are used in the central Section 5, where existence and uniqueness of these mathematical objects is discussed. The particular case of lattice equations is briefly reviewed in Section 6 and a few significant examples are proposed in Section 7.

2. Preliminary definitions

In the present work we will deal with applications $T_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}}: X \to X$ in a functional space X, defined by

$$T_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}}(f) := \sum_{n\in Z} \pi_n f(\alpha_n x + \beta_n)\ , \ \ f \in X. \tag{2.1}$$

Here α_n, β_n and π_n are real numbers. The space X where these applications act can be any of L^p, C_0^∞ (we will denote by K the support of the function) or the space of bounded non-decreasing (eventually continuous) functions, depending on the case at hand. In all cases we deal with univariate function on $\mathbb{R}$. Linear combinations as in the above equation arise in many applications. The first definition that we need to recall considers the application of Eq. (2.1) to refinable functions.

Definition 2.1 (Refinable functions)**.** A function ϕ is called refinable if it solves the fixed point problem:

$$\phi = T_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}}(\phi). \tag{2.2}$$

In the context of refinement equations the above is usually indicated as Functional Equation, see [5, equation (4.12)], and the vector $\{\pi_n\}$ is called a *mask*.

The most studied case is the one where the mask has a finite number of non-null entries. In this case, without loss of generality, one can write the previous equation as:

$$\phi(x) = \sum_{n=1}^{N} \pi_n \phi(\alpha_n x + \beta_n). \tag{2.3}$$

The existence of a solution of Eq. (2.3) is easily assessed in the space of distributions, under mild hypotheses. Much harder is to prove further regularity of this solution. This problem has been widely studied, as we will see later on. Since linearity implies that all multiples of a solution $\phi(x)$ are also solutions, a further condition, such as $\|\phi\| = 1$, must be imposed to obtain uniqueness.

Let us now define iterated function system (IFS) and their invariant measures [1–4]. These too are constructed as fixed points of an operator, that depends on a set of maps $\{S_n\}$ on a compact set K, typically, but not necessarily, included in **R**. Let $\mathcal{M}(K)$ be the space of Borel regular measures having bounded support K and mass equal to one, as in [30]. Also, let $\mathcal{S}$ be the operator from $\mathcal{M}(K)$ to itself defined as

$$\mathcal{S}(\sigma)(E) := \sum_{n=1}^{N} p_n \sigma(S_n^{-1}(E)), \tag{2.4}$$

for any σ–measurable set E.

Definition 2.2 (IFS Invariant Measures)**.** A measure μ with support in $K \subset \mathbb{R}$ is invariant for an IFS if there exists a collection of injective maps $S_n : K \to K,\ \ n = 1, \ldots, N$ and a set of probabilities $p_n,\ n = 1, \ldots, N$,

$$0 < p_n < 1, \ \sum_{n=1}^{N} p_n = 1 \tag{2.5}$$

so that

$$\mu = \mathcal{S}(\mu),\ i.e. \mu(E) = \sum_{n=1}^{N} p_n \mu(S_n^{-1}(E)) \quad \text{for every Borel set } E. \tag{2.6}$$

If the applications S_n are contractive, existence and uniqueness of the invariant measure is guaranteed [30, Theorem 4.4.1]. These conditions can be somehow relaxed [38] as we will see below.

Eq. (2.6) is sometimes called a balance equation and μ a balanced measure (that is therefore used as a synonymous of invariant). Consider now the transformations S_n. While in refinement equations almost invariably the affine maps $\alpha_n x + \beta_n$ has been considered, non-linear IFS maps S_n have also been studied [3]. Yet, the particularly simple case $S_n(x) := a_n x + b_n$ turns out to be quite versatile and useful. In this case, the operator in Eq. (2.6) will be labeled as $\mathcal{S}_{\mathbf{p},\mathbf{a},\mathbf{b}}$. The *inverse* maps S_n^{-1} correspond to the