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Weighted pseudo almost automorphic mild solutions for two-term fractional order differential equations

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ABSTRACT

Using the notion of Weyl fractional derivative, we study the existence and uniqueness of weighted pseudo almost automorphic mild solutions for a class of two-term fractional order abstract differential equation in the form

$$D_t^{\alpha} u'(t) + \mu D_t^{\beta} u(t) = Au(t) + D_t^{\alpha} f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha \leq \beta \leq 1, \quad \mu \geq 0$$

where A is a closed linear operator defined in a Banach space X and the forcing term f is S^p-weighted pseudo almost automorphic.

1. Introduction

Existence of almost automorphic, pseudo almost automorphic and weighted pseudo almost automorphic solutions is among the most attractive topics in qualitative theory of differential equations due to its significance and applications in physics, mechanics and mathematical biology. Therefore the study of the existence of almost automorphic and pseudo almost automorphic solutions of different kinds of differential equations has been considered recently.

In this paper we study the existence and uniqueness of weighted pseudo almost automorphic solutions for a class of two-term fractional differential equations of the form

$$D_t^{\alpha} u'(t) + \mu D_t^{\beta} u(t) = Au(t) + D_t^{\alpha} f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha \leq \beta \leq 1, \quad \mu \geq 0, \tag{1.1}$$

where A is a sectorial operator of angle $\beta\pi/2$, D^{γ} denotes the *Weyl fractional derivative* of order γ and the vector-valued function f is S^p-weighted pseudo almost automorphic.

The concept of weighted pseudo almost automorphic functions was introduced in 2009 by Blot, N'Guérékata, Mophou and Pennequin [8]. After that, Liu and Song [29] proved the existence and uniqueness of an weighted pseudo almost automorphic mild solution of the semilinear evolution equation $u'(t) = A(t)u(t) + f(t, u(t))$ in a Banach space under certain conditions of Acquistapace–Terreni type. See also [3]. In the paper [20] Ding, Long and N'Guérékata established an important composition

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theorem for the study of semilinear evolution equations by means of operator theoretical methods. In [1], Abbas introduces the notion of weighted pseudo almost automorphic sequences, studying several properties and including a composition result. See also [28] for a related result on sequences. Fatajou [22] proved the existence and uniqueness of weighted pseudo almost automorphic solutions for a class of partial functional differential equations in fading memory spaces using techniques of semigroup theory and exponential dichotomy. In the paper [9] Boukli-Hacene and Ezzinbi studied the existence and uniqueness of weighted pseudo almost automorphic solutions for some nonhomogeneous partial differential equations. Chang, Zhao and Nieto [13] studied existence and uniqueness of weighted pseudo almost automorphic solutions for a class of semilinear equations of the form $u'(t) = Au(t) + Bu(t) + F(t, u(h(t)))$, $t \in \mathbb{R}$ where A, B are closed operators defined on a Hilbert space H and F, h are functions satisfying certain appropriate conditions. In the paper [14], Chang, Zhao, Nieto and Liu established existence and uniqueness of weighted pseudo almost automorphic mild solutions for a class of neutral functional differential equations of the form $\frac{d}{dt}[u(t) + f(t, u(h_1(t)))] = Au(t) + g(t, u(h_2(t)))$ where A is the generator of an analytic semigroup and f, g, h_1, h_2 satisfy certain technical assumptions. Zitane and Bensouda [45] proved under suitable assumptions the existence and uniqueness of weighted pseudo almost automorphic solutions to a neutral delay integral equation of advanced type of the form $u(t) = \int_t^\infty f(u(h_1(s))) + Q(s, u(s), u(h_2(s)))C(t-s)ds + g(t)$ where h_1, h_2 generalize the delay functions and f, g, Q, C satisfy appropriate conditions. Weighted pseudo almost automorphic functions on time scales were introduced by Wang and Li [42]. Weighted pseudo almost automorphic solutions for non-autonomous stochastic evolution equations where studied by Chen and Lin [15]. Some recent results revealing general properties are due to Ding, Liang and Xiao [19]. See also the recent references [16,17,33–35] and [36] for related information.

The first result on weighted pseudo almost automorphic solutions for fractional differential equations is due to Mophou [37], who establishes the existence and uniqueness of weighted pseudo almost automorphic mild solutions to the semilinear fractional equation $D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t), Bu(t))$, where $1 < \alpha < 2$, the operator A is sectorial and B is bounded. Chang, Zhang and N'Guérékata [10] studied the same problem for the equation $D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t))$, but where the forcing term f is S^p-weighted pseudo almost automorphic. The notion of S^p-weighted pseudo almost automorphic functions was introduced in 2010 by Chang, N'Guérékata and Zhao in the article [12]. The use of S^p-weighted pseudo almost automorphic coefficients was also used by Mishra and Bahuguna [32] in the study of the problem $u'(t) + Au(t) = f(t, u(t), Ku(t))$ where $Ku(t) = \int_{-\infty}^t k(t-s)g(s, u(s))ds$. However, the study of weighted pseudo almost automorphic mild solutions for the class of two-term fractional differential Eqs. (1.1) with S^p-weighted pseudo almost automorphic coefficients remains open. The objective of this paper is to study this problem.

Eq. (1.1) in case $\mu = 0$ has been widely studied by several authors in the last decade, see [6,43] and references therein. In that case, it is known that the problem is strongly connected with integral equations of convolution type [40] and hence many properties can be deduced appealing to known results of abstract integral equations [6,44]. See also the references therein. Let us point out that a strong motivation for investigating such abstract fractional differential equations comes from physics [21,23]. Eq. (1.1) in case $\mu \neq 0$ arises as a consequence of recent investigations where a related class appears in connection with partial differential equations and Cauchy-time processes, a type of iterated stochastic processes (see [5], [38, Theorem 2.2] and [25]). For instance, in the border case $\alpha = 0$ and $0 < \beta \leq 1$ Eq. (1.1) corresponds to a fractional transport equation that in the radial case reproduces the early breakthrough of bromide observed in a radial tracer test conducted in a fractional granite aquifer [7]. Precise interplay between space-fractional and time-fractional order for partial differential equations was investigated in [25]. Very recently, the article [30] studied existence and uniqueness of solutions for the abstract Equation (1.1) in the special case $\alpha = \beta$ by means of an operator theoretical approach. Also, the article [41] studied the nonlinear two-term time fractional diffusion wave equation with $0 < \alpha < 1 - \beta$, $0 < \beta < 1$ and $A = \frac{d^2}{dx^2}$.

The paper is organized as follows. In Section 2, we recall some definitions, lemmas and preliminary results. In Section 3 we prove the remarkable fact that when using Weyl fractional derivative, the formula

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds$$

defines, for an appropriate choice of the family $\{S(t)\}_{t\geq 0}$ (see Definition 2.1), a true solution for the linear Eq. (1.1). Consequently, a consistent a well defined (mild) solution for the semilinear problem (1.1) is given by a fixed point of the equation

$$u(t) = \int_{-\infty}^t S(t-s)f(s, u(s))ds.$$

In Section 4, we prove the existence and uniqueness of weighted pseudo almost automorphic mild solutions to the two-term fractional order differential Eq. (1.1) when the forcing term f is S^p-weighted pseudo almost automorphic and Lipschitz continuous (Theorems 4.3 and 4.4) and not Lipschitz continuous (Theorem 4.5). We use an operator theoretical approach based on certain families of strongly continuous operators (Definition 2.1), and Banach fixed point theorem and Leray–Shauder alternative theorem as main tools. An example is given in Section 4, to illustrate the results obtained.

2. Preliminaries

Let us recall that a closed and densely defined operator A is said to be *ω-sectorial of angle θ* if there exist $\theta \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector

$$\omega + S_\theta := \left\{\omega + \lambda : \lambda \in \mathbb{C},\ |arg(\lambda)| < \frac{\pi}{2} + \theta\right\} \setminus \{\omega\}, \tag{2.1}$$