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Uniformly convergent hybrid numerical scheme for singularly perturbed delay parabolic convection–diffusion problems on Shishkin mesh

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ABSTRACT

This article studies the numerical solution of singularly perturbed delay parabolic convection–diffusion initial-boundary-value problems. Since the solution of these problems exhibit regular boundary layers in the spatial variable, we use the piecewise-uniform Shishkin mesh for the discretization of the domain in the spatial direction, and uniform mesh in the temporal direction. The time derivative is discretized by the implicit-Euler scheme and the spatial derivatives are discretized by the hybrid scheme. For the proposed scheme, the stability analysis is carried out, and parameter-uniform error estimates are derived. Numerical examples are presented to show the accuracy and efficiency of the proposed scheme.

1. Introduction

Ordinary and partial differential equations (ODEs and PDEs) play a crucial role in modeling biological systems, starting from the time of Verhulst, Lotka, and Volterra. The main aim of such models is to understand more and more complicated phenomena. It has been noticed that the ample diversity of dynamics observed in natural system cannot be captured by simple models. There are several approaches exist to deal with such difficulties. On one hand, one can construct larger systems of ODEs or PDEs. Such systems can be quite good at approximating observed behavior, but the difficulties with those cases come from the involvement of various parameters.

Another way of modeling is the inclusion of time delay terms in the differential equations. These delays/lags can represent gestation times, incubation periods, transport delays, or can simply lump complicated biological processes together, accounting only for the time required for these processes to occur. Such models have the advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes for a single system. Delay models are becoming more common, appearing in many branches of biological modeling. They have been used for describing several aspects of infectious diseases dynamics: primary infection, drug therapy and immune response to name a few. Delays have also appeared in the study of chemostat models, circadian rhythms, epidemiology, the respiratory system, tumor growth and neural networks.

In this article, we study the numerical solution of singularly perturbed delay partial differential equations (DPDEs), in which the highest-order derivative is multiplied by a small parameter $0 < \varepsilon \ll 1$ (called the singular perturbation parameter) and involving a delay term. When $\varepsilon \to 0$ in the equation, the solution exhibits a boundary layer, that is, the solution has steep gradients near the boundary of the domain. Singularly perturbed PDEs model physical problems for which the evolution depend in the

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present state of the system. In contrast the singularly perturbed DPDEs relate an unknown function to its derivatives by the past history.

The difference between singularly perturbed DPDEs and singular perturbed PDEs without time delay are:

(i) Singularly perturbed DPDEs can model the time-lag or after effect which cannot be done by conventional instantaneous PDEs.
(ii) If we replace $u(x, t-\tau)$ with the first few terms of the Taylor series expansion, then the solution of the approximated singularly perturbed PDE may behave quite differently from the original solution. Small lags can have large effects.
(iii) For singularly perturbed DPDEs, the initial condition is a function $\phi_b(x, t)$ defined on the interval $[-\tau, 0]$, whereas in the case of singularly perturbed PDEs, the initial value $\phi_b(x, 0)$ is given only on the x-axis, *i.e.*, at time $t = 0$.

To describe the motivation of DPDEs, in a better way, consider an automatically controlled furnace. The material strip to be heated is fed into the furnace by rollers whose speed is regulated by the speed controller. The furnace temperature is varied by means of a heater actuated by a heater controller. The control objective is to maintain a desired spatial temperature distribution in the incoming material, which is fed into the furnace by rollers, the speed of which is regulated by a speed controller. This may be accomplished by placing temperature transducers along the material strip. A computer uses the information from the transducers to generate the appropriate control signals for the heater and feed roller controllers. Owing to the possible presence of time delays in actuation, and in information transmission and processing, the control signals may be delayed in time. A simplified mathematical description of the overall control system may be given by

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \nu(g(u(x,t-\tau)))\frac{\partial u(x,t)}{\partial x} + c[f(u(x,t-\tau)) - u(x,t)],$$

defined on a one dimensional spatial domain $0 < x < 1$, where ν is the instantaneous material strip velocity depending on a prescribed spatial average of the time-delayed temperature distribution $u(x, t-\tau)$, and f represents a distributed temperature source function depending on $u(x, t-\tau)$. More details can be seen in [13].

1.1. Model problem

Let $\Omega = (0, 1)$, $D = \Omega \times (0, T]$, and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l and Γ_r are the left and the right sides of the rectangular domain D corresponding to $x = 0$ and $x = 1$, respectively, and $\Gamma_b = [0, 1] \times [-\tau, 0]$. In this paper, we consider the following class of singularly perturbed delay parabolic initial-boundary-value problems (IBVPs) with Dirichlet boundary conditions:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathcal{L}_\varepsilon\right)u(x,t) = -c(x,t)u(x,t-\tau) + f(x,t), & (x,t) \in D, \\ u(x,t) = \phi_b(x,t), & (x,t) \in \Gamma_b, \\ u(0,t) = \phi_l(t), & \text{on } \Gamma_l = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) = \phi_r(t), & \text{on } \Gamma_r = \{(1,t) : 0 \le t \le T\}, \end{cases} \tag{1.1}$$

where

$$\mathcal{L}_\varepsilon u(x,t) = -\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + b(x,t)u(x,t),$$

$0 < \varepsilon \ll 1$ and $\tau > 0$ are given constants, $a(x), b(x, t), c(x, t), f(x, t)$ on $\overline{D}$, and $\phi_l(t), \phi_r(t), \phi_b(x, t)$ on Γ, are sufficiently smooth and bounded functions that satisfy

$$a(x) \ge \alpha > 0, b(x,t) \ge 0, c(x,t) \ge \beta > 0 \text{ on } \overline{D}.$$

The terminal time T is assumed to satisfy the condition $T = k\tau$ for some positive integer k. The solution of the IBVP (1.1) exhibits regular boundary layer of width $O(\varepsilon)$ along $x = 1$.

The classical finite difference methods applied on uniform mesh, fail to provide satisfactory numerical solution for singularly perturbed differential equations, until we use unacceptably large number of mesh points in comparison with the perturbation parameter ε. This drawback motivates to develop the concept of ε-uniform numerical methods for singularly perturbed differential equations, *i.e.*, the order of convergence and the error constant are independent of ε.

Numerical methods for singularly perturbed differential equations (ODEs and PDEs) have been studied by many researchers over the last few decades. To cite a few: Natesan and Ramanujam [9] considered a boundary-value problem (BVP) for singularly perturbed ODE, and provided a booster method, which incorporates an asymptotic expansion into a numerical method and give higher-order accuracy. To obtain higher-order ε-uniform convergent numerical solution for singularly perturbed parabolic PDEs, Mukherjee and Natesan proposed the hybrid scheme in [6,7] and applied the Richardson extrapolation technique in [8].

Most of the articles available in the literature for numerical solution of singularly perturbed delay DEs are only ODEs. For example, Mohapatra and Natesan [5] devised an ε-uniform convergent scheme for singularly perturbed delay two-point BVPs, by using adaptive meshes obtained by equidistribution of a positive monitor function. However, the theory and numerical solution of singularly perturbed DPDEs are still at the primary stage. To obtain uniform convergent numerical solution to singularly perturbed DPDEs of reaction-diffusion type, classical finite difference schemes are applied on the Shishkin mesh by Ansari et. al. [1] and on layer-adapted mesh obtain through equidistribution by Gowrisankar and Natesan [2]. Also in [3], the authors solved the singularly perturbed DPDEs of convection–diffusion type by the upwind finite difference scheme on the Shishkin mesh.