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Adams–Simpson method for solving uncertain differential equation

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ABSTRACT

Uncertain differential equation is a type of differential equation driven by canonical Liu process. How to obtain the analytic solution of uncertain differential equation has always been a thorny problem. In order to solve uncertain differential equation, early researchers have proposed two numerical algorithms based on Euler method and Runge–Kutta method. This paper will design another numerical algorithm for solving uncertain differential equations via Adams–Simpson method. Meanwhile, some numerical experiments are given to illustrate the efficiency of the proposed numerical algorithm. Furthermore, this paper gives how to calculate the expected value, the inverse uncertainty distributions of the extreme value and the integral of the solution of uncertain differential equation with the aid of Adams–Simpson method.

1. Introduction

Probability theory, which was founded by Kolmogorov in 1933, has been widely used to model indeterminacy phenomena for a long time. A premise of applying probability theory is that the obtained probability distribution is close enough to the real frequency. However, due to some privacy or technological reasons, we usually have little or no sample data to estimate the probability distribution of a random variable. In this case, we have to invite some experts to give their belief degrees that each event will occur. A lot of surveys showed that human beings usually estimate a much wider range of values than the object actually takes (Liu [11]). This conservatism of human beings makes the belief degree deviate far from the frequency. Hence, it is inappropriate to treat the belief degree as a random variable and to model indeterminacy phenomena in this case by probability theory.

To deal with belief degree mathematically, Liu proposed uncertainty theory in 2007 [6] by uncertain measure. Then Liu [7] gave the concept of uncertain process to describe the evolution of an uncertain phenomenon, and designed a canonical Liu process [8] in contrast to Wiener process. Here, almost all the sample paths of a canonical Liu process are Lipschitz continuous, and Gao [3] weakened this condition and proposed the concept of semi-canonical process. Besides, Yao and Li [21] gave a special class of uncertain process called uncertain alternating renewal process in 2012. Based on the canonical Liu process, Liu [8] established uncertain calculus to deal with the integral and differential of an uncertain process. Following that, Chen [2] proposed an uncertain integral with respect to general Liu process.

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As an important tool to deal with uncertain dynamic system, uncertain differential equation driven by canonical Liu process, was first proposed by Liu [7] in 2008. In recent years, many researchers have done a lot of work about uncertain differential equation. In 2010, Chen and Liu [1] proved a sufficient condition for the existence and uniqueness of the solution of uncertain differential equation. Then Gao [4] gave a weaker condition. In 2014, Gao and Yao [5] presented two continuity theorems on solution of uncertain differential equation. As far as we know, stability analysis of solutions is one of the central problems in uncertain differential equation. The concept of stability for uncertain differential equation was first given by Liu [8], and some stability theorems were proved by Yao et al. [19]. Later on, different types of stability of uncertain differential equation were explored, such as stability in mean (Yao et al. [22]) and stability in moment (Sheng-Wang [16]).

The solution methods of uncertain differential equation also draw much attention from a lot of researchers. Chen and Liu [1] provided the analytic solution of a linear uncertain differential equation in 2010. After that, Liu [12] and Yao [20] gave the analytic solutions for some special classes of nonlinear uncertain differential equation. In fact, it is difficult to obtain the analytic solution of general uncertain differential equation. Researchers turn to look for the numerical solution of uncertain differential equation. In 2013, Yao and Chen [18] found a way to transfer the known uncertain differential equation into a family of associated ordinary differential equations. What's more, Yao and Chen [18] proposed famous Yao–Chen formula which can determine the inverse uncertainty distribution of the solution of an uncertain differential equation. Based on Yao–Chen formula, Yao [17] and Shen and Yao [14] presented Euler method and Runge–Kutta method to solve uncertain differential equation, respectively. Nowadays, uncertain differential equation has been applied to many areas especially in optimal control (Zhu [23] and Sheng et al. [15]) and finance (Liu [10] and Liu et al. [13]).

In this paper, we aim at providing another way to obtain the numerical solutions of uncertain differential equation via Adams–Simpson method. The remainder of this paper is organized as follows. The next section is intended to introduce some basic concepts and theorems in uncertainty theory and uncertain differential equation. Section 3 designs a numerical algorithm to solve uncertain differential equation by Adams–Simpson method, and presents one numerical example to show the comparison between the previous methods and Adams–Simpson method. In Section 4, Adams–Simpson method is applied for calculating the expected value, the inverse uncertainty distribution of extreme value and integral of solutions of uncertain differential equation. Finally, we make a brief conclusion in Section 5.

2. Preliminary

In this section, we introduce some basic concepts and theorems about uncertainty theory and uncertain differential equation, which are used throughout this paper.

2.1. Uncertainty theory

In order to provide a quantitative measurement that an uncertain phenomenon will occur, an axiomatic definition of uncertain measure is defined as follows.

Definition 2.1 (Liu [6])**.** Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ is called an *uncertain measure* if it satisfies the following axioms:

Axiom 1. (*Normality axiom*) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2. (*Duality axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3. (*Subadditivity axiom*) For every countable sequence of $\{\Lambda_i\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertainty space*, and each element Λ in $\mathcal{L}$ is called an *event*. Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by Liu [8] as follows:

1. Axiom 4. (*Product axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots,$ respectively.

In order to represent the quantities with uncertainty, an uncertain variable was proposed as a real valued function on an uncertainty space.

Definition 2.2 (Liu [6])**.** An *uncertain variable* ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.