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One order numerical scheme for forward–backward stochastic differential equations

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ABSTRACT

A one order numerical scheme based on the four step scheme developed by Ma et al. for the adapted solutions to a class of forward–backward stochastic differential equations is proposed and analyzed. For the decoupling quasilinear parabolic equations, a new kind of characteristics and finite difference method is used. While for the decoupled forward SDE, we use the Milstein scheme.

1. Introduction

Since the seminal work of Pardoux and Peng [5] in early 1990s, the theory of backward stochastic differential equations (BSDEs) and forward–backward stochastic differential equations (FBSDEs) has been found very useful in many applications fields such as stochastic optimizations and mathematical finance [1–3]. However, it is very difficult to obtain the analytic solution for the general BSDEs or FBSDEs, then, it is an interesting work to get an efficient numerical approximation solution. Many efforts have been made to the numerical solutions of the BSDEs or FBSDEs [9–11,15]. In the BSDEs case, various methods have been proposed, such as PDE method, random walk approximations, Malliavin calculus and Monte-Carlo method, the quantization method and etc. can be in [6,12–14,16,17].

In the case of coupled FBSDEs, the most of existing numerical results for FBSDEs: from the early work of Douglas et al. [7] to recent Milstein and Tretyakov [8] and Ma et al. [18], are based on the idea of the four step scheme developed by Ma et al. in [4]. Our aim is to give a one order numerical algorithm in the mean-square sense for the following FBSDEs:

$$X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s)dWs, \tag{1.1}$$

$$Y_t = g(X_T) + \int_t^T \widehat{b}(s, X_s, Y_s, Z_s)ds + \int_t^T Z_s dWs, \tag{1.2}$$

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where $t \in [0, T]$, X, Y, Z takes values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ and $b, \widehat{b}, \sigma, g$ are smooth functions with appropriate dimensions; $T > 0$ is an arbitrarily prescribed number which stands for the time duration. Due to the four step scheme from [4], the solution of (1.1) and (1.2) is connected with the Cauchy problem :

$$\theta_t + \frac{1}{2}\sigma^2(t,x,\theta)\theta_{xx} + b(t,x,\theta,z(t,x,\theta,\theta_x))\theta_x + \widehat{b}(t,x,\theta,z(t,x,\theta,\theta_x)) = 0, \tag{1.3}$$

$$\theta(T,x) = g(x), \tag{1.4}$$

$(t,x) \in (0,T)\times\mathbb{R}$. For the quasilinear parabolic Eqs. (1.3) and (1.4), a new kind of characteristics and finite difference method is used [19]. While for the decoupled forward SDE, we use the Milstein scheme.

The rest of the paper is organized as follows. In Section 2 we give the necessary preliminaries; we study the approximation for the quasilinear PDE and perform an error analysis in Section 3. In Section 4 we give a convergence result about the special FBSDE and give the result about the general FBSDE in Section 5. Then, in Section 6, we present a numerical example showing the result of the proposed method.

2. Preliminaries

Throughout this paper we assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space, on which is defined a d-dimensional Brownian motion $W = \{W(t) : t \geq 0\}$. Let $\mathcal{F}_t$ be the σ-field generated by W (*i.e.*, $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$) with usual P-augmentation so that it is right continuous and contains all the P-null sets in $\mathcal{F}$. For the sake of simplicity, in what follows we will consider only the case in which $n = m = d = 1$.

Definition 2.1. A triple of processes $(X, Y, Z) : [0,T]\times\Omega \to \mathbb{R}^3$ is called an L^2-adapted solution of the FBSDE (1.1) and (1.2) if it is $\mathcal{F}_t-$ adapted and square integrable and is such that it satisfies (1.1) and (1.2) almost surely.

For any $T > 0$ and $\alpha \in (0, 1)$, we define $C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\mathbb{R})$ to be the space of all functions $\phi(t, x)$ which are differentiable in t and twice differentiable in x which ϕ_t and ϕ_{xx} being $\frac{\alpha}{2}$-and α-Hölder continuous in $(t,x) \in [0,T]\times\mathbb{R}$. The norm in $C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times R)$ is defined by

$$\begin{aligned}\|\phi\|_{1,2,\alpha} &= \|\phi\|_C + \|\phi_t\|_C + \|\phi_x\|_C + \|\phi_{xx}\|_C \\ &\quad + \sup_{(t,x)\neq(t',x')} \frac{|\phi_t(t,x) - \phi_t(t',x')| + |\phi_{xx}(t,x) - \phi_{xx}(t',x')|}{(|x-x'|^2 + |t-t'|)^{\frac{\alpha}{2}}},\end{aligned}$$

where $\|\cdot\|_C$ is the usual sup-norm on the set $[0,T]\times\mathbb{R}$.

We will make use of the following standing assumptions throughout the paper.

(A1) The functions $b, \widehat{b}, \sigma$ are continuously differentiable in t and twice continuously differentiable in x, y, z. If we denote any one of these functions generically by ψ , then there exists a constant $\alpha \in (0, 1)$, such that for fixed y and z, $\psi(\cdot,\cdot,y,z) \in C^{1+\frac{\alpha}{2},2+\alpha}$. Furthermore, for some $L > 0$,

$$\|\psi(\cdot,\cdot,y,z)\|_{1,2,\alpha} \leq L, \quad \forall (y,z) \in \mathbb{R}^2.$$

(A2) The function σ satisfies

$$\mu \leq \sigma(t,x,y) \leq C, \quad \forall (t,x,y) \in [0,T]\times\mathbb{R}^2,$$

where $0 \leq \mu \leq C$ are two constants.

(A3) The function g belongs boundedly to $C^{4+\alpha}$ for some $\alpha \in (0, 1)$.

The four step scheme.

Step 1. Define a function $z : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ by

$$z(t,x,y,p) = -p\sigma(t,x,y), \qquad \forall (t,x,y,p). \tag{2.1}$$

Step 2. Solve the following quasilinear parabolic equation for $\theta(t,x)$ in $C^{1+\frac{\alpha}{2},2+\alpha}$, for some $0 < \alpha < 1$:

$$\begin{aligned}&\theta_t + \frac{1}{2}\sigma^2(t,x,\theta)\theta_{xx} + b(t,x,\theta,z(t,x,\theta,\theta_x))\theta_x + \widehat{b}(t,x,\theta,z(t,x,\theta,\theta_x)) = 0, \\ &(t,x) \in (0,T)\times\mathbb{R}, \\ &\theta(T,x) = g(x), \quad x \in \mathbb{R}.\end{aligned} \tag{2.2}$$

Step 3. Using θ and z, solve the forward SDE

$$X_t = x + \int_0^t \widetilde{b}(s,X_s)ds + \int_0^t \widetilde{\sigma}(s,X_s)dWs, \tag{2.3}$$

where $\widetilde{b}(s,X_s) = b(t,x,\theta(t,x),z(t,x,\theta(t,x),\theta_x(t,x)))$ and $\widetilde{\sigma}(s,X_s) = \sigma(t,x,\theta(t,x))$.