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Adaptive least squares finite integration method for higher-dimensional singular perturbation problems with multiple boundary layers

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ABSTRACT

Based on the recently developed finite integration method for solving one-dimensional partial differential equation, we extend in this paper the method by using the technique of least squares to tackle higher-dimensional singular perturbation problems with multiple boundary layers. Theoretical convergence and numerical stability tests indicate that, even with the most simple numerical trapezoidal integration rule, the proposed method provides a stable, efficient, and highly accurate approximate solutions to the singular perturbation problems. An adaptive scheme on the refinement of integration points is also devised to better capture the stiff boundary layers. Illustrative examples are given in both 1D and 2D with comparison among some existing numerical methods.

1. Introduction

Singular perturbation problems arise in many physical modeling processes of fluid flows. Due to the stiffness and complexity of the boundary layers exerted by the singularly perturbed equations, it is very difficult to obtain exact solutions for these equations. The rapid advancement of computing technology allows the simulation of the boundary layers through seeking numerical approximate solutions to the mathematical models. During the last decades, these singular perturbation problems have been extensively studied by many researchers using various kinds of numerical methods [1–5] among which the finite difference method (FDM) and finite element method (FEM) are mostly well established. Despite their many attractive features, the FDM and FEM are mesh-dependent which require an appropriate generation of good mesh for accurate and stable approximation to the solution. The generation of good mesh, however, is not a trivial task for problems defined under irregular domain. For stiff problems with boundary layers to be investigated in this paper, some kinds of adaptive schemes on mesh refinement have to be established. In fact, when the singularly perturbed equation is perturbed at a very small parameter ϵ attached to a higher derivative term in the equation, the solution normally forms a layer where the solution derivatives change dramatically within a very narrow interval. This causes most standard computational methods fail to produce satisfactory approximation. It had been proven by Miller et al. [6] that the numerical methods composed of upwind- and central-difference operators on uniform mesh were defective for solving the singularly perturbed differential equations. In fact, Miller et al. had proven that the numerical

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oscillation errors are unbounded as $\epsilon \to 0$ on uniform meshes. A good review of the numerical methods for these singularly perturbed problems can be found in [7,8]. Recent advances in numerical approximation for the singularly perturbed problems are concentrated on performing the computations on specially designed meshes. Miller et al. [6] reported that the Shishkin meshes [9] provide a better numerical results by using upwind- and central-difference schemes. Similar result was also obtained by Sun and Stynes [10] in applying the FEM using piecewise polynomial test/trial functions on the Shishkin meshes.

The use of multiquadric collocation method (MQCM), which is a kind of kernel-based approximation methods, was first proposed by Hon [11] to solve a one-dimensional singular perturbation problem with one boundary layer. The meshless advantage of MQCM was shown to provide an efficient and accurate approximation of the boundary layer. Ling and Trummer [12] later combined the kernel-based approximation method with an adaptive scheme to obtain an improved highly accurate solution for solving the one-dimensional boundary layer problem. For multiple boundary layers problems, Zhang [13] gave a theoretical approach from which numerical approximation can be constructed. In this paper, we extend the recently developed finite integration method (FIM) [14,15] for solving higher-dimensional singularly perturbed equations with multiple boundary layers. Unlike FDM which uses finite quotient formula, the FIM uses numerical quadratic integration rule and hence avoids the well-known roundoff-discretization error problem in using FDM. Both theoretical and numerical results given in this paper show that, even with the most simple numerical trapezoidal integration rule, the FIM provides a very stable, efficient, and highly accurate approximation to the solutions of these higher-dimensional singular perturbation problems. Numerical comparisons with the MQCM and FDM with Shishkin grids show that the FIM achieves better accuracy at lower computational cost.

This paper is organized as follow. The FIM is briefly introduced in Section 2. The formulation of the methodology to solve a singularly perturbed equation with multiple boundary layers is given in Section 3. Theoretical convergence analysis is discussed in Section 4. Section 5 demonstrates the improved accuracy of the method in solving the singularly perturbed equation with one single layer by comparing with the exact solution and some recent works. Numerical stability test is performed to verify the convergence order in one-dimensional case. An adaptive scheme is proposed in Section 6 with illustrative example given in Section 7 on its superior stability and higher convergence to solve the perturbed equation with double layers. In Section 8, we combine the least squares technique with FIM to give a new least-squares finite integration method (LSFIM) to solve a two-dimensional singular perturbed problem with stiff boundary layer. Stability and convergence order are derived with numerical examples verified for two-dimensional case. Some concluding remarks with suggested future works are provided in the final conclusion section.

2. Finite integration method

Based on our recently developed finite integration method for solving one-dimensional partial differential equations [15] and the technique of least squares, we will extend in this section the method to solve higher-dimensional singular perturbation problems with multiple boundary layers. For simplicity, we first consider 1D case and define $F^{(1)}(x)$ to be the integration of an integrable function $f(x)$

$$F^{(1)}(x) = \int_a^x f(\xi)\, d\xi, \tag{1}$$

where $x \in [a, b] \subset R$.

Assume a partition $a = x_1 < x_1 < \ldots < x_N = b$ of the interval $[a, b]$ into subintervals, $[a, b] = \cup_{i=1}^{N-1}[x_i, x_{i+1}]$, the above integration when $x = x_k$ can be written as

$$F^{(1)}(x_k) = \int_a^{x_k} f(\xi)\, d\xi = \sum_{i=1}^{k-1} \int_{x_i}^{x_{i+1}} f(\xi)\, d\xi. \tag{2}$$

In each subinterval, the use of numerical trapezoidal rule to the integration gives:

$$\begin{aligned}\int_{x_i}^{x_{i+1}} f(\xi)\, d\xi &= \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) - \frac{f''(\xi_i)}{6}(x_{i+1} - x_i)^3 \\ &\triangleq \frac{\Delta_i}{2} f_i + \frac{\Delta_i}{2} f_{i+1} - \frac{f''(\xi_i)}{6}\Delta_i^3,\end{aligned} \tag{3}$$

where $f_i = f(x_i)$, $f_{i+1} = f(x_{i+1})$, $\Delta_i = x_{i+1} - x_i$, $i = 1, 2, \ldots, k-1$ and $\xi_i \in (x_{i-1}, x_i)$. Combining Eqs. (2) and (3), we have

$$\begin{aligned}F^{(1)}(x_k) &= \sum_{i=1}^{k-1}\left(\frac{\Delta_i}{2} f_i + \frac{\Delta_i}{2} f_{i+1}\right) - \sum_{i=1}^{k-1}\frac{f''(\xi_i)}{6}\Delta_i^3 \\ &= \frac{\Delta_1}{2} f_1 + \sum_{i=2}^{k-1}\frac{\Delta_{i-1} + \Delta_i}{2} f_i + \frac{\Delta_{k-1}}{2} f_k - \sum_{i=1}^{k-1}\frac{f''(\xi_i)}{6}\Delta_i^3 \\ &\triangleq \sum_{i=1}^{k} a_{ki}^{(1)} f_i - \sum_{i=1}^{k-1}\frac{f''(\xi_i)}{6}\Delta_i^3,\end{aligned} \tag{4}$$