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Oscillation properties for the equation of the relativistic quantum theory

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A B S T R A C T

In this paper we consider the boundary value problem for the canonical one-dimensional Dirac system and investigate the oscillatory properties of the eigenvector-functions of this problem. We find the number of zeros of components of the eigenvector-functions in dependence on the parameters of boundary conditions and give a refinement the asymptotic formula for the eigenvalues of the considered problem.

1. Introduction

We consider the following Dirac differential equation

$$Bw'(x) - P(x)w(x) = \lambda w(x), \ \ 0 < x < \pi, \tag{1.1}$$

with the boundary conditions $U(w) = \binom{U_1(w)}{U_2(w)} = 0$ given by

$$U_1(w) := (\sin\alpha, \ \cos\alpha)\, w(0) = v(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$

$$U_2(w) := (\sin\beta, \ \cos\beta)\, w(\pi) = v(\pi)\cos\beta + u(\pi)\sin\beta = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}, \quad w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

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$\lambda \in \mathbb{C}$ is a spectral parameter, $p(x)$ and $r(x)$ are real valued, continuous functions on the interval $[0, \pi]$, α and β are real constants; moreover $0 \leq \alpha, \beta < \pi$.

Eq. (1.1) is equivalent to the system of two consistent first-order ordinary differential equations

$$\begin{aligned} v' - \{\lambda + p(x)\}u = 0, \\ u' + \{\lambda + r(x)\}v = 0. \end{aligned} \tag{1.4}$$

If the boundary value problem (1.1)–(1.3) has a non-trivial solution

$$w(x, \lambda) = \begin{pmatrix} u(x, \lambda) \\ v(x, \lambda) \end{pmatrix}$$

for some $\lambda = \tilde{\lambda}$, then the number $\tilde{\lambda}$ is called an eigenvalue, and the corresponding solution $w(x, \tilde{\lambda})$ is called the eigenvector-function.

For the case in which $p(x) = V(x) + m,\ r(x) = V(x) - m$, where $V(x)$ is a potential function, and m is the mass of a particle, the system (1.1) is known in relativistic quantum theory as a stationary one-dimensional Dirac system.

Ever since its invention in 1929 the Dirac equation has played a fundamental role in various areas of modern physics and mathematics. Relativistic quantum mechanics is the historical origin of the Dirac equation and has become a fixed part of the education of theoretical physicists. The rich mathematical structure of the Dirac equation has attracted a lot of interest and many surprising results were obtained which were included in a systematic exposition of the Dirac theory [18,26].

The oscillatory properties for the eigenfunctions of the linear ordinary differential operators have a long history. In 1836, Sturm ([24], [[18], Chapter 1]), using an analytic approach, studied these properties for a second-order self-adjoint linear differential operator. Thereafter, the oscillation properties of the eigenfunctions of the Sturm–Liouville problem completely studied in various formulations with different methods (see, e.g. [8,10,12,14]). Using a different approach based on the analysis of the kernel of the integral equation associated with the Sturm–Liouville problem, Kellog [14] obtained the same oscillatory properties of Sturm. Later this approach was extended by many authors (e.g., see [2,8,11,13,16,17]) to study the spectral and the oscillatory properties of a class of higher-order boundary value problems.

For the first time, in 1988, McLaughlin showed that knowledge of nodal zeros of eigenfunctions can determine the potential function of the Sturm–Liouville operator up to a constant [19](see also [9]). Unlike a typical inverse problem, in which potential is determined from spectral data (usually, the spectral data contains two sets of eigenvalues, or one set of eigenvalues and norming constants), it is called an inverse nodal problem. For the one-dimensional Dirac system (1.1)–(1.3) the inverse nodal problem was studied in [27]. In [28] the authors deal with an inverse nodal problem of reconstructing the Dirac system with the spectral parameter in the boundary conditions and prove that a set of nodal points of one of the components of the eigenfunctions uniquely determines all the parameters of the boundary conditions and the coefficients of the Dirac equations.

The basic and comprehensive results (except the oscillation properties) about the Dirac operator were given in [18]. It is known (see [18], Chapter 1, Section 11) that eigenvalues of the boundary value problem (1.1)–(1.3) are real, algebraically simple and the values range from $-\infty$ to $+\infty$ and can be numerated in increasing order. In [28] studied the oscillatory properties of the eigenvector-functions of (1.1)–(1.3) for sufficiently large $|n|$, but, unfortunately, found there the number of zeros of components of eigenvector-functions is not accurate. In [15] and [28] (see also the references in [28]) studied the oscillation properties of eigenvector-functions of the Dirac system with a spectral parameter in the boundary conditions. It should be noted that these studies did not found the exact number of zeros of the components of the eigenvector-function corresponding to the nth ($n \in \mathbb{Z}$) eigenvalue. In [25] the author developed an extension of the results of [29] to study the oscillation theory for one-dimensional Dirac operators with separated boundary conditions, where found the number of zeros of the Wronskian of solves of the Dirac equation (in the weak sense). In this work as an application, the author establishes finiteness of the number of eigenvalues in essential spectral gaps of perturbed periodic Dirac operators. In [23] the authors developed relative oscillation theory for one-dimensional Dirac operators which, rather than measuring the spectrum of one single operator, measures the difference between the spectra of two different operators. This is done by replacing zeros of solutions of one operator by weighted zeros of Wronskians of solutions of two different operators. They show that a Sturm-type comparison theorem still holds in this situation and demonstrate how this can be used to investigate the number of eigenvalues in essential spectral gaps. In [7] particular cases of the oscillation theory of the problem (1.1)–(1.3) are considered in detail. In [3] we study general characteristic of the location of the eigenvalues on the real axis and we develop an extension of the classical Sturm theory [24] (see also [10,12]), to study the oscillatory properties for the eigenvector-functions of problem (1.1)–(1.3). But we cannot completely study oscillatory properties of components of the eigenvector-functions of this problem.

The above reasoning show that the problem on the number of zeros of components of the vector-functions of (1.1)–(1.3) is subject to a detailed study which is devoted the present paper.

In Section 2 we develop an extension of the classical theory presented in [[4], Chapter 8, Section 4], to study the properties of the functions $v/u,\ u/v$ and the angular function $\theta = \tan^{-1}(v/u)$ (satisfying the initial conditions at $x = 0$) in respect of their dependence on x and λ which is played important role by the discussion of the problem on number of zeros of u and v. In Section 3 (which consists of six sections) we mainly study the oscillatory behavior of the eigenvector-functions of the system (1.1)–(1.3). Here we prove the main result (Theorem 3.1) consisting of two parts, in which in the first part is given the numbering of the eigenvalues in ascending order on the real axis depending on values of the angular function θ at $x = \pi$ and in a suitable interpretation, in the second part is given the number of zeros of components of the corresponding eigenvector-functions. Furthermore, we give refinement of the asymptotic formula (11.18) from [[18], Chapter 1].