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A novel numerical method to determine the algebraic multiplicity of nonlinear eigenvalues

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ABSTRACT

We generalize the algebraic multiplicity of the eigenvalues of nonlinear eigenvalue problems (NEPs) to the rational form and give the extension of the argument principle. In addition, we propose a novel numerical method to determine the algebraic multiplicity of the eigenvalues of the NEPs in a given region by the contour integral method. Finally, some numerical experiments are reported to illustrate the effectiveness of our method.

1. Introduction

We consider computing the eigenpairs (λ, x), where $\lambda \in C$ and $x \in C^m$ $(x \neq 0)$, of the nonlinear eigenvalue problems (NEPs) of the form

$$T(\lambda)x = 0, \tag{1.1}$$

where $T(\lambda) \in C^{m\times m}$ is a holomorphic matrix-valued function with respect to λ. The NEPs arise in a number of various applications, such as numerical simulation of quantum dots, aircraft structure design, the material electronic structure calculation, calculation of electromagnetic field, and so on. Due to wide applications of the NEPs, we urgently need to develop efficient numerical methods for the NEPs. However, the Schur factorization of $T(\lambda)$ does not exist, so it is a great challenge to design the numerical algorithms for solving the NEPs.

More than ten years ago, Mehrmann and Voss [1] wrote an excellent review of the most frequently used methods for solving the NEPs. The NEP (1.1) may be reformulated as a nonlinear equation through computing the characteristic polynomial $f(\lambda) := \det(T(\lambda))$ or a system of nonlinear equations through adding a normalization equation, and then the Newton method can be applied to solve the so-obtained nonlinear equations. Using QR decomposition, Kublanovskaya [2] proposed a Newton method for solving the NEP (1.1). Jain and Singhal [3] pointed out that Kublanovskaya's algorithm did work in spite of some errors in the derivation and provided quadratic convergence. Li [4] gave sufficient conditions to guarantee the existence of a differentiable QR decomposition and modified the Newton method based on the QR decomposition. Besides, Li [5] presented an incomplete QR decomposition to compute multiple eigenvalues of the NEPs. Dai and Bai [6] developed the smooth LU decomposition of analytic matrix-valued matrices and proposed a Newton method for computing multiple nonlinear eigenvalues of the NEPs. Based on the singular value decomposition (SVD), Guo et al. [7] presented a Newton method for solving the NEP (1.1) and developed a nonequivalence deflation technique.

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For the above-described methods to compute multiple eigenvalues, the algebraic multiplicity of the eigenvalues is very critical, and it is also very important for the deflation technique. So it is necessary to consider the algebraic multiplicity and present an algorithm to determine the algebraic multiplicity of the eigenvalues of the NEPs. In this paper, we generalize the algebraic multiplicity of the eigenvalues of the NEPs to the rational form and present an extension of the argument principle. In addition, we provide a novel numerical method to determine the algebraic multiplicity of the eigenvalues of the NEPs in a given region.

The remainder of the paper is organized as follows. In Section 2, we generalize the algebraic multiplicity of the eigenvalues of the NEPs to the rational form and give the extension of the argument principle. In Section 3, we propose a novel method to determine the algebraic multiplicity of the eigenvalues of the NEPs using a contour integral method. Section 4 is devoted to some numerical experiments. Finally, in Section 5 we use some conclusions to end this paper.

Throughout this paper, we shall adopt the following notations: $tr(A)$ and $det(A)$ denote the trace and the determinant of matrix A, respectively. ι denotes the imaginary unit of a complex number.

2. Argument principle and its extension

It is well known that the argument principle is an important theorem in complex analysis, see, e.g., [8], which can be stated as follows.

Theorem 2.1. *Assume that $f(z)$ is a meromorphic function on and inside some closed contour Γ, $f(z)$ is analytic on Γ and $f(z)$ has no zeros or poles on Γ, then*

$$\frac{1}{2\pi\iota}\oint_\Gamma \frac{f'(z)}{f(z)}dz = N(f,\Gamma) - P(f,\Gamma),$$

where $N(f,\Gamma)$ and $P(f,\Gamma)$ denote the number of zeros and poles of $f(z)$ inside the contour Γ, respectively, counted according to their multiplicity.

The above theorem leads to the following corollary directly.

Corollary 2.1. *Assume that $f(z)$ is a meromorphic function on and inside some closed contour Γ and $f(z)$ is analytic on Γ. If $f(z)$ has no zeros on Γ and no poles on and inside Γ, then*

$$\frac{1}{2\pi\iota}\oint_\Gamma \frac{f'(z)}{f(z)}dz = N(f,\Gamma),$$

where $N(f,\Gamma)$ is the number of zeros of $f(z)$ inside the contour Γ, counted according to their multiplicity.

Now, we consider extending the argument principle.

For the NEP (1.1), let $f(z) = det(T(z))$, and then $f(z)$ is an analytic function with respect to z. It is obvious that z is an eigenvalue of the NEP (1.1) if and only if z is a root of the nonlinear equation $det(T(z)) = 0$. When $T(z)$ is a matrix polynomial, $f(z)$ is a polynomial of z.

Assume that $T(z)$ has n different eigenvalues $z_1, z_2, \dots, z_n$ inside a contour Γ. If the algebraic multiplicity of every eigenvalue z_i is a positive integer, denoted as m_i, then $f(z)$ can be written as $f(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}\dots(z-z_n)^{m_n}g(z)$, where $g(z)$ is analytic and has no any zeros or poles on and inside Γ.

In the following, we will generalize the algebraic multiplicity of the eigenvalue of the NEPs to the rational form, i.e., the value of m_i is not always a positive integer, and sometimes it may be a fraction.

Definition 2.1. Assume that $f(z)$ is a meromorphic function on and inside some closed contour Γ, $f(z)$ has n different zeros $z_1, z_2, \dots, z_n$ inside Γ and no any poles on and inside Γ, and

$$f(z) = (z-z_1)^{\alpha_1}(z-z_2)^{\alpha_2}\dots(z-z_n)^{\alpha_n}g(z), \tag{2.1}$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the degree of the zeros $z_1, z_2, \dots, z_n$, respectively, each of α_i is a rational number, $g(z)$ is analytic and has no any zeros on and inside Γ. Then z_i is called as the fulcrum of $f(z)$ whose order is $\alpha_i (i = 1, \dots, n)$.

Remark 2.1. For polynomial eigenvalue problems including quadratic eigenvalue problems, if $f(z)$ in Definition 2.1 is the characteristic polynomial of the NEP (1.1), then z_i and α_i are the eigenvalue and the corresponding algebraic multiplicity of $T(z)$, respectively.

The following example provides an intuitive illustration on the idea of Definition 2.1.

Example 2.1. For the NEP (1.1), where

$$T(z) = \begin{pmatrix} (z-1)^{\frac{1}{2}} & & \\ & (z-2)^{\frac{1}{3}} & \\ & & (z-3)^{\frac{1}{4}} \end{pmatrix},$$

then $f(z) = (z-1)^{\frac{1}{2}}(z-2)^{\frac{1}{3}}(z-3)^{\frac{1}{4}}$.

It is easy to know that 1, 2 and 3 are the eigenvalues of $T(z)$. According to Definition 2.1, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ are the algebraic multiplicities of the eigenvalues 1, 2 and 3, respectively.

Based on Definition 2.1, we can generalize the argument principle as follows.