Applied Mathematics and Computation 271 (2015) 365–374

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On limit cycles bifurcating from the infinity in discontinuous piecewise linear differential systems

Márcio R.A. Gouveia [a], Jaume Llibre [b], Douglas D. Novaes [c,*]

[a] Departamento de Matemática, IBILCE–UNESP, Rua C. Colombo, 2265, CEP 15054-000 S. J. Rio Preto, São Paulo, Brazil
[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[c] Departamento de Matemática, Universidade Estadual de Campinas, Rua Sérgio Baruque de Holanda, 651, Cidade Universitária Zeferino Vaz, CEP 13083-859, Campinas, São Paulo, Brazil

ARTICLE INFO

MSC:
34A36
34C29
37G15

Keywords:
Discontinuous differential system
Piecewise linear differential system
Infinite periodic orbit
Limit cycle
Displacement function

ABSTRACT

In this paper we consider the linear differential center $(\dot{x}, \dot{y}) = (-y, x)$ perturbed inside the class of all discontinuous piecewise linear differential systems with two zones separated by the straight line $y = 0$. Using the Bendixson transformation we provide sufficient conditions to ensure the existence of a crossing limit cycle coming purely from the infinity. We also study the displacement function for a class of discontinuous piecewise smooth differential system.

1. Introduction and statement of the main result

Establishing sufficient conditions for the existence of periodic solutions, specially *limit cycles*, i.e. periodic solutions isolated in the set of all periodic solutions, is one of the main problems of the qualitative theory of planar differential systems. A classical way to produce and to study limit cycles is perturbing the periodic solutions of a *center*, which is a point having a neighborhood, except itself, filled by periodic solutions. This problem for smooth differential systems in the plane has been studied intensively, either for limit cycles bifurcating from finite periodic solutions (see, for instance, hundreds of references in the book [4]), or for limit cycles bifurcating from the infinity (see, for instance, [7,21,22]). In [8] the authors studied the bifurcation of a limit cycle from the infinity for a non-smooth but continuous piecewise differential system. Nevertheless, as far as we know, there are no studies of limit cycles bifurcating from the infinity for discontinuous differential systems. Accordingly, this is the objective of the present paper: to study limit cycles coming from the infinity for a class of discontinuous differential systems.

Recently the theory of discontinuous differential systems has been strongly developed, with growing importance at the frontier between mathematics, physics, engineering, and the life sciences. Interest stems particularly from discontinuous dynamical models (see, for instance, [15,20]). Additionally, the existence of periodic solutions gives important information about the dynamics of these models, as the presence of oscillatory motions (see, for instance, the book of Andronov, Vitt, and Khaikin [2], and the book of Minorski [16], both classical and important works on physical oscillatory phenomena), which represents one of the main source of physical motivation for this study. There are some works dealing with limit cycles bifurcating from finite periodic solutions in discontinuous differential systems, see, for instance, [3,9,10,12–14,17,18]. To the best of our knowledge, this paper is the first work dealing with limit cycles bifurcating from the infinity for discontinuous systems.

* Corresponding author. Tel.: +55 1935215921; fax: +55 1935216094.
E-mail addresses: maralves@ibilce.unesp.br (M.R.A. Gouveia), jllibre@mat.uab.cat (J. Llibre), ddnovaes@ime.unicamp.br (D.D. Novaes).

http://dx.doi.org/10.1016/j.amc.2015.09.022
0096-3003/

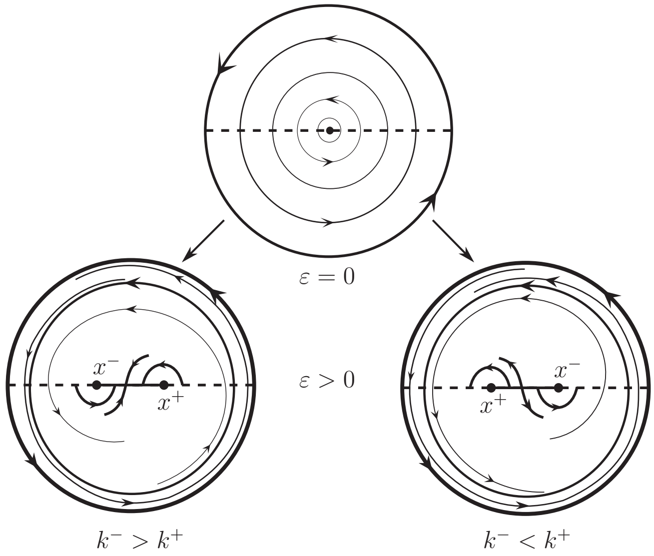

Fig. 1. Behavior of the periodic orbits (in the Poincaré disc) of system (1) satisfying the hypotheses (*iii*) of Theorem 1. For $\varepsilon = 0$ system (1) is a center. For $\varepsilon > 0$ and $k^- > k^+$ system (1) has an unstable limit cycle, near to the infinity, and a sliding region in the discontinuous set. For $\varepsilon > 0$ and $k^- < k^+$ system (1) has a stable limit cycle, near to the infinity, and a scape region in the discontinuous set. The diameters crossing the discs represents the discontinuous set Σ, where the dashed lines are the crossing regions Σ^c and the continuous lines are the sliding Σ^s and escaping Σ^e regions when $k^- > k^+$ and $k^- < k^+$, respectively. The points $(x^{\pm}, 0)$ are called *invisible folds* of the vector fields $F^{\pm}$. For a definition of Poincaré disc, see, for instance, Chapter 5 of [5].

1.1. Limit cycles from the infinity

In this paper we consider the linear differential center $(\dot{x}, \dot{y}) = (-y, x)$ perturbed inside the class of all discontinuous piecewise linear differential systems with two zones separated by the straight line $\Sigma = \{y = 0\}$, that is

$$\dot{x} = \begin{cases} -y + \varepsilon(a^+ + b^+x + c^+y) + \varepsilon^2(\alpha^+ + \beta^+x + \gamma^+y) & \text{if } y > 0, \\ -y + \varepsilon(a^- + b^-x + c^-y) + \varepsilon^2(\alpha^- + \beta^-x + \gamma^-y) & \text{if } y < 0, \end{cases}$$
$$\dot{y} = \begin{cases} x + \varepsilon(k^+ + m^+x + n^+y) + \varepsilon^2(\kappa^+ + \mu^+x + \nu^+y) & \text{if } y > 0, \\ x + \varepsilon(k^- + m^-x + n^-y) + \varepsilon^2(\kappa^- + \mu^-x + \nu^-y) & \text{if } y < 0. \end{cases} \tag{1}$$

The dot denotes derivative with respect to the time t.

Here we are interested in the study of *crossing limit cycles*, that is limit cycles that do not contain sliding segments and tangential points. Roughly speaking, we say that a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ comes *purely* from the infinity if $||(x(t, \varepsilon), y(t, \varepsilon))|| \to \infty$ when $\varepsilon \to 0$ for every $t \in \mathbb{R}$. In [3] the authors found, for a class of discontinuous piecewise linear perturbation of the linear center, conditions for the existence of periodic solutions $(x(t, \varepsilon), y(t, \varepsilon))$ such that $||(x(t, \varepsilon), y(t, \varepsilon))|| \to \lambda$ when $\varepsilon \to 0$, moreover the level $\lambda \in \mathbb{R}$ can be taken as bigger as wanted. Nevertheless, since $||(x(t, \varepsilon), y(t, \varepsilon))|| = \lambda + \mathcal{O}(\varepsilon)$, taking $\lambda \to \infty$ is not sufficient to prove the existence of a periodic solution coming purely from the infinity for some perturbation. Our first main result in this paper is about the crossing limit cycles of system (1). For completeness, in the statements (*i*) and (*ii*) we study the birth (and the uniqueness) of a crossing limit cycle that bifurcates from the origin and from the periodic solutions of the linear center (that is, system (1) when $\varepsilon = 0$), respectively. The crossing limit cycle bifurcating purely from the infinity is studied in the statement (*iii*).

Theorem 1. *Regarding to system (1) the following statements hold.*

(i) *If* $(b^+ + b^- + n^+ + n^-)(k^- - k^+) > 0$ *then, for* $|\varepsilon| \neq 0$ *sufficiently small, there exists a unique crossing limit cycle* $(x_1(t, \varepsilon), y_1(t, \varepsilon))$ *of system (1) which is stable (resp. unstable) provided that* $b^+ + b^- + n^+ + n^- < 0$ *(resp.* $b^+ + b^- + n^+ + n^- > 0$*). Moreover* $||(x_1(t, \varepsilon), y_1(t, \varepsilon))|| \to 4(k^- - k^+)/(\pi(b^+ + b^- + n^+ + n^-)) > 0$ *when* $\varepsilon \to 0$ *for every* $t \in \mathbb{R}$.

(ii) *If* $k^- - k^+ = 0$ *and* $(b^+ + b^- + n^+ + n^-)\big(4a^-(b^- + n^-) - 4(k^+(m^- - m^+) + a^+(b^+ + n^+) + \kappa^+ - \kappa^-) - k^+(b^+ + b^- + n^+ + n^-)\big) > 0$ *then, for* $\varepsilon > 0$ *sufficiently small, there exists a unique crossing limit cycle* $(x_2(t, \varepsilon), y_2(t, \varepsilon))$ *of system (1) which is stable (resp. unstable) provided that* $b^+ + b^- + n^+ + n^- < 0$ *(resp.* $b^+ + b^- + n^+ + n^- > 0$*). Moreover* $||(x_2(t, \varepsilon), y_2(t, \varepsilon))|| \to 0$ *when* $\varepsilon \to 0$ *for every* $t \in \mathbb{R}$.

(iii) *If* $b^+ + b^- + n^+ + n^- = 0$ *and* $(k^- - k^+)\big((b^+ + n^+)(c^- - c^+ + m^+ - m^-) - 2(\beta^+ + \beta^+ + \nu^+ + \nu^-)\big) < 0$ *then, for* $\varepsilon > 0$ *sufficiently small, there exists a unique crossing limit cycle* $(x_3(t, \varepsilon), y_3(t, \varepsilon))$ *of system (1) which is stable (resp. unstable) provided that* $k^- < k^+$ *(resp.* $k^- > k^+$*). Moreover this limit cycle comes purely from the infinity, that is* $||(x_3(t, \varepsilon), y_3(t, \varepsilon))|| \to \infty$ *when* $\varepsilon \to 0$ *for every* $t \in \mathbb{R}$ *(see Fig. 1).*