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Eigenvalue analysis of a generalized indefinite block triangular preconditioner for generalized saddle point problems[☆]

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ABSTRACT

In this paper, we consider a generalized indefinite block triangular preconditioner for the generalized saddle point problems. The eigenvalue analysis of the preconditioned matrix is given, which generalizes the existing results in the literature. Some corrections to the theoretical or numerical results in the previously published works are also presented. Numerical experiments are provided to confirm the bounds for the eigenvalues of the preconditioned matrix, and illustrate the efficiency of the preconditioned GMRES.

1. Introduction

Consider the generalized saddle point problem

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n} (m \le n)$ is of full row rank, and $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, and $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are two column vectors. Systems of the form (1.1) frequently arise in computational science and engineering applications such as computational fluid dynamics [8], constrained optimization [20], parameter identification [12], mixed finite element approximation of elliptic PDEs [5,14,18], and others. Krylov subspace methods together with various preconditioners are recommended for the saddle point problems in large scale. In the passed decades, there are lots of work on this topic, including block diagonal preconditioner [9,15,16,18], block triangular preconditioner [7,10,13,19,22,24], HSS preconditioner [1,2,4], constraint preconditioner [3,6,11], matrix splitting preconditioner [21] and so on.

Recently, Simoncini [19] and Cao [7], respectively, proposed the block triangular preconditioners $P_- = \left[\begin{smallmatrix} \hat{A} & B^T \\ O & -\hat{C} \end{smallmatrix}\right]$ and $P_+ = \left[\begin{smallmatrix} \hat{A} & B^T \\ O & \hat{C} \end{smallmatrix}\right]$, together with a Krylov subspace iterative method for solving (1.1), where $\hat{A} \in \mathbb{R}^{n\times n}$ and $\hat{C} \in \mathbb{R}^{m\times m}$ are symmetric positive definite. Spectral properties of the preconditioned matrices $\mathcal{A}P_-^{-1}$ and $\mathcal{A}P_+^{-1}$ were investigated in [7,19]. By introducing the parameter ω to the submatrix B^T of the block triangular preconditioners P_- and P_+, Jiang et al. considered in [10] two parameterized block triangular preconditioners as follows

$$\hat{P}_- = \begin{bmatrix} \hat{A} & \omega B^T \\ O & -\hat{C} \end{bmatrix} \text{ and } \hat{P}_+ = \begin{bmatrix} \hat{A} & \omega B^T \\ O & \hat{C} \end{bmatrix}.$$

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Similarly, Wu et al. [22] applied the following parameterized block triangular preconditioner

$$\tilde{P} = \begin{bmatrix} \hat{A} & B^T \\ O & j\hat{C} \end{bmatrix}, (j \neq 0)$$

for the linear system (1.1). A special case of $\tilde{P}$ when $j < 0$ was studied in [23–25]. The ideas of [10] and [22] were combined in [13], resulting in the generalized block triangular preconditioner

$$\hat{P} = \begin{bmatrix} \hat{A} & \omega B^T \\ O & j\hat{C} \end{bmatrix}. \tag{1.2}$$

If $j > 0$, all the eigenvalues of the preconditioned matrix $\mathcal{A}\hat{P}^{-1}$ are real. The eigenvalue estimates for $\mathcal{A}\hat{P}^{-1}$ were provided, which improve some theoretical results appeared in the previous works such as [7,10]. When $j < 0$, the bounds for the real part of all the eigenvalues of $\mathcal{A}\hat{P}^{-1}$ were mainly discussed. Indeed, the eigenvalues of $\mathcal{A}\hat{P}^{-1}$ with negative j may be real or complex, and the eigenvalue distribution of the real or complex eigenvalues was not fully investigated in [13].

The aim of this paper is to further consider the eigenvalue analysis of the generalized block triangular preconditioner for the saddle point problems, and to complement the theoretical results in [13]. We focus on the generalized indefinite block triangular preconditioner for solving (1.1), that is

$$P = \begin{bmatrix} \hat{A} & \omega B^T \\ O & -j\hat{C} \end{bmatrix}, \omega > 0 \text{ and } j > 0. \tag{1.3}$$

The eigenvalue estimates for the real and imaginary parts of all the complex eigenvalues of the preconditioned matrix $\mathcal{A}P^{-1}$ are provided, which generalize the theoretical results in [6,25]. Some inaccuracies in [25] are pointed out, and correct results are presented. We also give the bound for the real eigenvalues of the preconditioned matrix that is not touched in [13]. For the special case when $C = 0$, we provide the more refined bounds on the complex eigenvalues of $\mathcal{A}P^{-1}$. Choosing appropriate ω in the block triangular preconditioner P, only real eigenvalues exist for $\mathcal{A}P^{-1}$. The sufficient conditions for this case are presented, and the tight bounds for all the real eigenvalues are given, see Corollary 2.4. Numerical experiments of two examples are given. We present the bounds for the real or complex eigenvalues of the preconditioned matrix $\mathcal{A}P^{-1}$ to confirm the theoretical analysis. We also illustrate the efficiency of the preconditioned GMRES with the generalized block triangular preconditioner, based upon the eigenvalue analysis presented in Section 2. Finally, we give the conclusions.

2. Eigenvalue analysis of the preconditioned matrix $\mathcal{A}P^{-1}$

Let us consider an eigenvalue λ and the corresponding eigenvector $[u; v]$ of $\mathcal{A}P^{-1}$, that is,

$$\mathcal{A}P^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$

Denote $[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}] = P^{-1}[\begin{smallmatrix} u \\ v \end{smallmatrix}]$, the above eigenvalue problem can be equivalently rewritten as

$$\mathcal{A}\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \lambda P \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \tag{2.1}$$

Let $\hat{D} = [\begin{smallmatrix} \hat{A}^{\frac{1}{2}} & O \\ O & \sqrt{j}\hat{C}^{\frac{1}{2}} \end{smallmatrix}]$ and $[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}] = \hat{D}^{-1}[\begin{smallmatrix} \hat{u} \\ \hat{v} \end{smallmatrix}]$, then (2.1) is equivalent to

$$\hat{D}^{-1}\mathcal{A}\hat{D}^{-1}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \lambda \hat{D}^{-1}P\hat{D}^{-1}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}$$

or

$$\begin{bmatrix} \hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}} & \frac{1}{\sqrt{j}}\hat{A}^{-\frac{1}{2}}B^T\hat{C}^{-\frac{1}{2}} \\ \frac{1}{\sqrt{j}}\hat{C}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}} & -\frac{1}{j}\hat{C}^{-\frac{1}{2}}C\hat{C}^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \lambda \begin{bmatrix} I & \frac{\omega}{\sqrt{j}}\hat{A}^{-\frac{1}{2}}B^T\hat{C}^{-\frac{1}{2}} \\ O & -I \end{bmatrix}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}. \tag{2.2}$$

Let $\tilde{A} = \hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}}$, $\tilde{B} = \frac{1}{\sqrt{j}}\hat{C}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}}$, and $\tilde{C} = \frac{1}{j}\hat{C}^{-\frac{1}{2}}C\hat{C}^{-\frac{1}{2}}$, we can simplify (2.2) as the following generalized eigenvalue problem

$$\begin{bmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & -\tilde{C} \end{bmatrix}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \lambda \begin{bmatrix} I & \omega\tilde{B}^T \\ O & -I \end{bmatrix}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}. \tag{2.3}$$

In the following discussion, for a complex eigenvalue λ, we use $\mathcal{R}(\lambda)$ and $\mathcal{I}(\lambda)$ to denote its real and imaginary parts, respectively. By Lemma 1 in [6], we have the following lemma about the matrix $(\tilde{A} - \lambda I)^{-1}$ that is required for our analysis later.