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Boundedness of certain sets of Lagrange multipliers in vector optimization

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ABSTRACT

In this paper, we establish Lagrange multiplier rules in terms of Michel–Penot subdifferential for nonsmooth vector optimization problem. A constraint qualification or regularity condition in terms of Michel–Penot subdifferential is given and under this regularity condition the boundedness of certain sets of Lagrange multipliers are discussed.

1. Introduction

In the present article, we consider the following vector optimization problem (VP):

$$\left.\begin{aligned} \min\ f(x) &= (f_1(x), \dots, f_p(x)) \\ \text{subject to } g_i(x) &\le 0;\ i = 1, 2, \dots, m, \\ h_j(x) &= 0;\ j = 1, 2, \dots, n, \end{aligned}\right\} \tag{VP}$$

where $f := (f_1, f_2, \dots, f_p) : X \to \mathbb{R}^p, g_i : X \to \mathbb{R}(i = 1, \dots, m)$ and $h_j : X \to \mathbb{R}(j = 1, \dots, n)$ are functions on a real Banach space X. For $p = 1$ (denote $f := f_0$), vector optimization problem (VP) reduces to scalar optimization problem (SP). Throughout the paper, in the case of vector optimization problem (VP), we consider all functions involved are locally Lipschitz not necessarily differentiable if otherwise stated. For simplicity we denote $P = \{1, 2, \dots, p\}$, $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$. Let us denote the feasible set of (VP) by Ω, defined as

$$\Omega = \left\{x \in X | g_i(x) \le 0, h_j(x) = 0;\ i \in M,\ j \in N\right\}.$$

For any feasible solution $x_\circ$ of (VP), we denote

$$I(x_\circ) = \left\{i | g_i(x_\circ) = 0;\ i \in M\right\},$$

the index set of active or binding constraints. Denote by $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ the nonnegative orthant and interior of nonnegative orthant of $\mathbb{R}^n$ respectively. We introduce two commonly used notions of vector minimum.

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A feasible solution $x_\circ$ for (VP) is said to be an efficient point or the Pareto minimum of (VP) if there exists no other feasible solution x such that

$$f(x) - f(x_\circ) \in -\mathbb{R}^p_+ \backslash \{0\}$$

and the feasible solution $x_\circ$ for (VP) is said to be a weak efficient point or the weak Pareto minimum of (VP) if there exists no other feasible solution x such that

$$f(x) - f(x_\circ) \in -\mathbb{R}^p_{++}$$

In the past several decades many authors concerned to replace the usual gradient by certain generalized gradients under locally Lipschitz assumptions, e.g. see Rockafellar [11], Clarke [3], Michel and Penot [10], Ioffe [6,7] and Treiman [12]. One of the most important issues for optimality conditions concerns the size of the subdifferential. It is known that the Clarke generalized gradient may have extraneous subgradients for a differentiable function (but not strictly differentiable), which are not the usual derivative. The Michel–Penot (MP) subdifferential is the smallest one, and hence we aim to use a multiplier rule in terms of the MP-subdifferential established by Ioffe [7]. Ioffe showed that the Clarke generalized multiplier rule can be sharpened by replacing the Clarke generalized gradient by the MP-subdifferential, which coincides with usual derivative in the case of Gâteaux differentiability. The multiplier rules in terms of other bigger generalized gradients follow immediately.

In (VP) Maeda [8] showed the necessary conditions for efficiency under the same constraint qualifications as used in scalar case results to undesirable situation. Lagrange multipliers associated with the vector-valued objective function, namely, some of the multipliers may be equal to zero. To avoid this situation and to get positive Lagrange multipliers, Maeda [8] involved the components of objective function in constraint qualification, which will technically be referred to as the regularity condition.

Gauvin [5] first demonstrated that the Mangasarian–Fromovitz constraint qualification (see[9]) is a necessary and sufficient condition for a mathematical programming problem (scalar optimization) with smooth data to have a bounded set of Lagrange multipliers. Dutta and Lalitha [4] have investigated the set of Lagrange multipliers in the context of vector optimization. They identified certain subsets of KKT multipliers, which are non-empty and bounded under Basic Regularity Conditions (BRC) for (VP). Dutta–Lalitha [4] have considered the problem (VP) with smooth data and also when the underlying data are locally Lipschitz which need not be differentiable. Chandra et al. [2] studied the boundedness of the sets of proper KKT multipliers when (VP) consists only of inequality constraints for the case of an efficient point. Thus from scalar optimization point of view our results are more general than Gauvin [5], Chandra et al. [2] and Dutta–Lalitha [4].

In this article we consider both equality and inequality constraints as well as both efficient point and weak efficient point for nonsmooth case. This paper is organized as follows. Section 2 consists of basic tools and results of nonsmooth analysis, to be used in the succeeding sections. Section 3 is devoted to the constraint qualifications and regularity conditions. We use these constraint qualifications and regularity conditions to establish KKT type necessary optimality conditions, for efficient points of (VP) with the Lipschitz data, in terms of MP-subdifferential. In this section, we will also discuss the boundedness of set of proper KKT multipliers and provide an example to show that if (WBRC),which is analogous to (BRC), fails then the certain subsets of set of KKT multipliers are unbounded.

2. Preliminaries

In this section, we reproduce some basic definitions associated with nonsmooth analysis, which will be used in succeeding sections. We use the following notations, for any Banach space X we denote dual of X by X^*, equipped with weak-star topology w^*. For any subset S of X, the notation coS represents convex hull of S.

We reproduce some definitions of Michel–Penot derivative and subdifferential given as follows.

Definition 2.1 (see [13]). (Michel–Penot subdifferential): Let X be any Banach space, $x_\circ \in X$ and $f : X \to \mathbb{R}$ be any function. Then the Michel–Penot directional derivative of f at the point $x_\circ$ in the direction $v \in X$, denoted by $f^\diamond(x_\circ; v)$, is given by

$$f^\diamond(x_\circ; v) = \sup_{w \in X} \limsup_{t \downarrow 0} \frac{f(x_\circ + tv + tw) - f(x_\circ + tw)}{t}$$

and the Michel–Penot subdifferential of f at $x_\circ$ is given by

$$\partial^\diamond f(x_\circ) = \{x^* \in X^* : f^\diamond(x_\circ; v) \geq \langle x^*, v\rangle,\ \forall v \in X\},$$

it is known (see [10]) that when a function f is Gâteaux differentiable at $x_\circ$, $\partial^\diamond f(x_\circ) = \{\nabla f(x_\circ)\}$.

The following properties of the Michel–Penot directional derivatives and Michel–Penot subdifferentials will be useful in the sequel.

Proposition 2.1 (see [13]). *Let X be a Banach space, $x_\circ \in X$ and $f : X \to \mathbb{R}$ be the locally Lipschitz function at $x_\circ$, then the following hold:*

(i) *The function $v \longmapsto f^\diamond(x_\circ; v)$ is finite, positively homogeneous and subadditive on X.*
(ii) *As a function of v, $f^\diamond(x_\circ; v)$ is Lipschitz continuous with Lipschitz constant L_f.*
(iii) *$\partial^\diamond f(x_\circ)$ is a nonempty, convex, weak*- compact subset of X^* and $\|x^*\| \leq L_f$ for every $x^* \in \partial^\diamond f(x_\circ)$, one has $f^\diamond(x_\circ; v) = max\left\{\langle \xi^*, v\rangle | \xi^* \in \partial^\diamond f(x_\circ)\right\}$.*

We now consider only the Vector optimization problem (VP) with locally Lipschitz data.