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Periodic analytic approximate solutions for the Mathieu equation

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ABSTRACT

We propose two methods to find analytic periodic approximations intended for differential equations of Hill type. Here, we apply these methods on the simplest case of the Mathieu equation. The former has been inspired in the harmonic balance method and designed to find, making use on a given algebraic function, analytic approximations for the critical values and their corresponding periodic solutions of the Mathieu differential equation. What is new is that these solutions are valid for all values of the equation parameter q, no matter how large. The second one uses truncations of Fourier series and has connections with the least squares method.

1. Introduction

The harmonic balance method [1–4] has been designed in order to obtain analytic approximations to periodic solutions for certain type of ordinary differential equations, linear or not. In this paper, we introduce an improved version of it and we test it in the Mathieu equation, in order to compare its efficiency with respect other methods and more specifically of the least square method. Henceforth, we shall denote it as the "Modified Harmonic Balance Method".

The Mathieu equation is the simplest non-trivial type of Hill equation. This is a second order linear differential equation of the following type:

$$\frac{d^2y(x)}{dx^2} + \left(A_0 + \sum_{n=1}^{\infty} A_n \cos(2nx) + \sum_{m=1}^{\infty} B_m \sin(2mx) \right) y(x) = 0. \tag{1}$$

This expression comes from a general linear second order equation of the type

$$y''(x) + f(x)\, y(x) = 0, \tag{2}$$

in which we have spanned $f(x)$ into Fourier series. Eq. (1), which is often called the general Hill equation, is intractable in general terms. In order to study tractable approximations to (1), we must truncate the series involved in the equation. In general, one chooses $B_1 = B_2 = \cdots = 0$. Then, we say that the Hill equation is of order $n-th$, if $A_n \neq 0$ and $A_{n+1} = A_{n+2} = \cdots = 0$. Hill equations

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of first and second order are known as Mathieu and Whittaker–Hill equations, respectively [5–8]. Further order Hill equations may be of interest in physics [9].

It is customary to write the Mathieu equation on the following form:

$$y''(x) + (r - 2q\cos(2x))y(x) = 0, \tag{3}$$

where $A_0 = r$, which is often called the *characteristic value* or eigenvalue, has to be determined through given boundary conditions, and $A_1 = -2q$, which is a fixed data. The solution $y(x)$ for a given r is usually called the eigenfunction. Generally speaking, one looks for periodic solutions of (3), with boundary conditions at two finite fixed points, say 0 and p, given on the form $y(0) = y(p)$ and $y'(0) = y'(p)$, where p is the period. This is certainly a particular case of a Sturm–Liouville problem.

The properties of the Mathieu equation (3) has been extensively studied [10]. It has been established the existence of four countable sets of values for the characteristic value r, for which there exists periodic solutions, with period either π or 2π. These four sets are associated to solutions that admit a series expansion on either even functions of the type $\cos(2k)x$ or $\cos(2k+1)x$, or odd functions of the form $\sin(2kx)$ or $\sin(2k+1)x$. Obviously, if $q = 0$, the characteristic values are $r = m^2$, $m = 1, 2, 3, \dots$ and the solutions are $\cos(mx)$ and $\sin(mx)$. For $q \neq 0$, there is only one periodic solution for each characteristic value, being the second solution non-periodic. See also [11].

The Sturm–Liouville problem associated with (1) has been solved by either methods based in Fourier series techniques [12–15] or by Taylor expansions [1,10]. These methods require of numerical computations and they are usually valid for $|q| < 1$ only. It seems that it would be desirable to have a method for solutions for large values of $|q|$.

Analytic approximate methods are usually one of the best tools to attack this kind of Sturm–Liouville problems. In order to approximate periodic solutions, the harmonic balance method has been widely used [1,10], either for linear or non-linear ordinary differential equations. Here, we do not consider the non-linear case and for the Mathieu equation, we propose an analytic approximate method that provides both the critical value r as well as the approximate periodic solution, which is valid even for large values of $|q|$. In few words, we obtain r as solutions of some algebraic equations. No integrations, Taylor expansions nor complex manipulations with matrices are needed. Our method is conceptually simple and has been inspired in the harmonic balance method.

We must stress that the modified harmonic balance method is in principle applicable to other Hill equations beyond the Mathieu equation and eventually other type of equations admitting periodic solutions. We have focused our calculations in the Mathieu equation to test our preliminary results. Our modification avoids completely certain complications of the standard harmonic balance, as presented in [1], like the need of solving non-homogeneous differential equations with increasing complexity when more terms are included. In our modified harmonic balance, we just need to solve algebraic equations.

We should compare the results obtained by this modified harmonic balance with those got by more traditional methods, like the least square method. The results are essentially similar with degree of accuracy in both methods. Nevertheless, ours have the advantages of being simpler and easier to use by a computer and the least square method is more complicated to be practically implemented. For instance, the polynomial giving the approximate characteristic values is much simpler in our modified harmonic balance than in the least square method, as shown in Section 4.1. This means, in particular, that the degree of algebraic equations is definitively smaller in our method as compared with the least squares. Note that the latter also gives spurious imaginary parts for roots.

The present article is organized as follows: In Section 2, we introduce our modification of the harmonic balance method and give some numerical results. In Section 3, we apply the method to a Mathieu equation with a purely imaginary parameter. In Section 4, we adapt the least square method to our situation. The results obtained by least squares are similar, although one needs both more precision and terms to obtain the same accuracy. In a limit sense, we show that the modified harmonic balance and the least square method are equivalent. We have written a section with concluding remarks plus an appendix on the relation between periodicity and parity of solutions for the general Hill equation.

2. A modification on the harmonic balance method

Since our method has been inspired in the usual harmonic balance method, it could be interesting to recall the latter before a discussion of our ideas. Then, we first add here an introductory subsection with some comments on the harmonic balance, which will be also appropriate to justify further analysis. Then, our method will be introduced in the second subsection.

However, one should be aware of one important difference between the harmonic balance and our method. While the former relies on Taylor expansions on q, ours does not.

2.1. The usual harmonic balance method

We are looking for periodic solutions for (2) determined by the boundary conditions $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$. In this case, the harmonic balance method, proposes the following type of solutions [1]:

$$r_m = m^2 + \sum_{k=1}^{\infty} \alpha_k q^k, \qquad Y_m(x) = \cos mx + \sum_{k=1}^{\infty} q^k\, c_k(x). \tag{4}$$

Note that if $q = 0$, then, $r_m = m^2$ and $Y_m(x) = \cos mx$ give an even periodic solution, so that the Ansatz for $Y_m(x)$ is intended to construct even solutions. A similar Ansatz can be provided to construct odd solutions by replacing $\cos mx$ by $\sin mx$. The