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New iterative technique for solving a system of nonlinear equations

Muhammad Aslam Noor[a,*], Muhammad Waseem[b], Khalida Inayat Noor[a]

[a] Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan
[b] Department of Mathematics, COMSATS Institute of Information Technology, Sahiwal, Pakistan

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ABSTRACT

Various problems of pure and applied sciences can be studied in the unified frame work of the system of nonlinear equations. In this paper, a new family of iterative methods for solving a system of nonlinear equations is developed by using a new decomposition technique. The convergence of the new methods is proved. Efficiency index of the proposed methods is discussed and compared with some other well-known methods. The upper bounds of the error and the radius of convergence of the methods are also found. For the implementation and performance of the new methods, the combustion problem, streering problem and Van der Pol equation are solved and the results are compared with some existing methods. Several new iterative methods are derived from the general iterative scheme. Using the ideas and techniques of this paper, one may be able to suggest and investigate a wide class of iterative methods for solving the system of nonlinear equations. This is another direction of future research.

1. Introduction

We consider the system of nonlinear equations of the type

$$\begin{aligned} f_1(x_1,x_2,\ldots,x_n) &= 0,\\ f_2(x_1,x_2,\ldots,x_n) &= 0,\\ &\cdots\\ f_n(x_1,x_2,\ldots,x_n) &= 0, \end{aligned}$$

where each function f_i, $i = 1, 2, \cdots\cdots, n$, maps a vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)^t$ of the n-dimensional space $\mathbb{R}^n$ to the real line $\mathbb{R}$. The above system of n nonlinear equations in n unknowns can also be represented by defining a function $\mathbf{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ as

$$\mathbf{F}(\mathbf{X}) = (f_1(\mathbf{X}),\ f_2(\mathbf{X}), \cdots\cdots, f_n(\mathbf{X}))^t.$$

which we usually write $\mathbf{F}(\mathbf{X}) = \mathbf{0}$, where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be nonlinear mapping from n -dimensional real linear space $\mathbb{R}^n$ into itself. The components f_i, $i = 1, 2, \ldots, n$, are the coordinate functions of $\mathbf{F}$.

Moré [33] presented a collection of nonlinear model problems, most of which are phrased in terms of $\mathbf{F}(\mathbf{X}) = \mathbf{0}$. Applications of the system of nonlinear equations in neurophysiology, chemical equilibrium problem, kinematics, combustion problem

* Corresponding author. Tel.: +92 3454027532.
E-mail addresses: noormaslam@hotmail.com, noormaslam@gmail.com (M.A. Noor), sattarmwaseem@gmail.com (M. Waseem), khalidanoor@hotmail.com (K.I. Noor).

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and economics modeling problem are considered in [22]. Awawdeh [5] and Tsoulos and Stavrakoudis [54] solved $\mathbf{F}(\mathbf{X})=\mathbf{0}$, by considering reactor and steering problems. Lin et al. [32] discussed the applications of $\mathbf{F}(\mathbf{X})=\mathbf{0}$, in transport theory.

To find the solution of the system of nonlinear equations, n-dimensional Newton method [49] is one of the fundamental tools and is quadratically convergent. Other different variants of Newton method have been developed for solving $\mathbf{F}(\mathbf{X})=\mathbf{0}$, using various techniques. Hueso et al. [27,28] have used Taylor polynomials, Babolian et al. [8], Cordero et al. [12], Darvishi and Barati [14], Kaya and El-Sayed [30] and Vahidi et al. [55] have applied Adomian decomposition technique [2,3] for solving the system of nonlinear equations. Abbasbandy [1], Darvishi and Barati [16] and Jafari and Daftardar-Gejji [29] used different modifications of Adomian decomposition method, Özel [50] have applied Noor decomposition technique [35,38] and Golbabai and Javidi [19,20] applied homotopy perturbation method [24] for solving $\mathbf{F}(\mathbf{X})=\mathbf{0}$. Awawdeh [5] and Hosseini and Hosseini [26] have used homotopy analysis method [31], Babajee and Dauhoo [6], Babajee et al. [7], Cordero and Torregrosa [10,11] , Darvishi and Barati [15,16], Frontini and Sormani [18] and Gordji et al. [21] have applied quadrature formulas [9,51] to develop iterative methods for system of nonlinear equations. Some methods such as Halley [4,23] method, Chebyshev methods [23] and the method of Awawdeh [5] despite of their cubic convergence are considered less practical from a computational point of view, since these methods involve the costly second-order Fréchet derivative. The methods using the second-order derivative are more complicated as compared to the methods using the first-order derivative.

In this paper, a new decomposition technique is used to develop a new family of iterative methods for solving $\mathbf{F}(\mathbf{X})=\mathbf{0}$, which is quite different from Adomian decomposition method [2,3]. In the implementation of the Adomian's method, one has to calculate the derivatives of the so-called Adomian polynomials, which is itself a difficult problem. To overcome the difficulty of Adomian's method, we use the decomposition of Daftardar-Gejji and Jafari [13], as developed by Noor and Noor [37] and Noor et al. [44], to suggest several new iterative methods for solving the system of nonlinear equations. This new decomposition does not involve the high-order differentials of the function and is very simple as compared with the Adomian decomposition method. He [25] has suggested that the system of nonlinear equations can be written as a coupled system of equations. This idea has been used by Noor [36] to develop some iterative methods for solving nonlinear equations. Noor et al. [41–48] have used the idea of coupled system of equations with the new decomposition technique to develop several iterative methods for solving nonlinear equations. In this paper, the technique and ideas of [48] are extended for solving the system of nonlinear equations. The convergence of the new methods is proved. Several numerical examples are given to illustrate the efficiency and performance of these iterative methods. Comparison of these methods with the other similar methods is also provided. We would like to mention that the methods proposed in this paper involves only the first-order Fréchet derivative of the function. The interested readers are encouraged to explore the applications of the suggested methods for solving other complicated problems. For more details, see [43,45–47] and the references therein.

2. Iterative methods

In this section, we develop a family of iterative methods for solving the system of nonlinear equations by using a new decomposition technique.

We consider the system of nonlinear equations

$$\mathbf{F}(\mathbf{X})=\mathbf{0}, \tag{1}$$

where $\mathbf{X}=(x_1,x_2,\dots,x_n)^t\in\mathbb{R}^n$. Assume that $\alpha\in\mathbb{R}^n$ be a zero of the system of nonlinear Eq. (1), and $\boldsymbol{\Omega}$ be an initial guess sufficiently close to $\boldsymbol{\alpha}$. By using the technique of [41,42], one can rewrite the system of nonlinear Eq. (1) as a coupled system of equations (see also [25]):

$$\mathbf{F}(\boldsymbol{\Omega})+\left[\sum_{i=1}^{p}w_i\mathbf{F}'(\boldsymbol{\Omega}+\tau_i(\mathbf{X}-\boldsymbol{\Omega}))\right](\mathbf{X}-\boldsymbol{\Omega})+\mathbf{G}(\mathbf{X})=\mathbf{0}, \tag{2}$$

$$\mathbf{G}(\mathbf{X})=\mathbf{F}(\mathbf{X})-\mathbf{F}(\boldsymbol{\Omega})-\left[\sum_{i=1}^{p}w_i\mathbf{F}'(\boldsymbol{\Omega}+\tau_i(\mathbf{X}-\boldsymbol{\Omega}))\right](\mathbf{X}-\boldsymbol{\Omega}), \tag{3}$$

where τ_i are knots in [0, 1] and w_i are the weights which verify the consistency condition

$$\sum_{i=1}^{p}w_i=1, \tag{4}$$

and $\boldsymbol{\Omega}$ is the initial approximation for the zero of (1).

Eq. (2) can rewritten in the following form:

$$\mathbf{X}=\boldsymbol{\Omega}-\left[\sum_{i=1}^{p}w_i\mathbf{F}'(\boldsymbol{\Omega}+\tau_i(\mathbf{X}-\boldsymbol{\Omega}))\right]^{-1}(\mathbf{F}(\boldsymbol{\Omega})+\mathbf{G}(\mathbf{X}))=\mathbf{C}+\mathbf{N}(\mathbf{X}), \tag{5}$$

where

$$\mathbf{C}=\boldsymbol{\Omega}, \tag{6}$$