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Some inequalities for the trigamma function in terms of the digamma function

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ABSTRACT

In the paper, the authors establish three kinds of inequalities for the trigamma function in terms of the exponential function to powers of the digamma function. These newly established inequalities extend some known results. The method in the paper utilizes some facts from the asymptotic theory and is a natural way to solve problems for approximating some quantities for large values of the variable.

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1. Motivations and main results

The classical Euler gamma function may be defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t.$$

The logarithmic derivative of the gamma function $\Gamma(z)$ is denoted by

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}\, z}[\ln \Gamma(z)] = \frac{\Gamma'(z)}{\Gamma(z)}$$

and called the digamma function. The derivatives $\psi'(z)$ and $\psi''(z)$ are called the trigamma and tetragamma functions, respectively. As a whole, the functions $\psi^{(k)}(z)$ for $k \in \{0\} \cup \mathbb{N}$ are called the polygamma functions. These functions are widely used in theoretical and practical problems in all branches of mathematical science. Consequently, many mathematicians were preoccupied to establish new results about the gamma function, polygamma functions, and other related functions.

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1.1. The first main result

In 2007, Alzer and Batir [3, Corollary] discovered that the double inequality

$$\sqrt{2\pi}\, x^x \exp\left[-x - \frac{1}{2}\psi(x+\alpha)\right] < \Gamma(x) < \sqrt{2\pi}\, x^x \exp\left[-x - \frac{1}{2}\psi(x+\beta)\right] \tag{1.1}$$

holds for $x > 0$ if and only if $\alpha \ge \frac{1}{3}$ and $\beta \le 0$. For information on generalizations of the results of the paper [3], please refer to [12,36] and references cited therein. Motivated by the double inequality (1.1), Mortici [30] proposed the asymptotic formula

$$\Gamma(x) \sim \sqrt{2\pi}\, e^{-b}(x+b)^x \exp\left[-x - \frac{1}{2}\psi(x+c)\right], \quad x \to \infty$$

and then determined the optimal values of parameters b, c in such a way that this convergence is the fastest possible.

In 2004, Batir [4, Theorem 2.1] presented the double inequality

$$\Gamma(c)\exp[\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1] \le \Gamma(x) \le \Gamma(c)\exp\left\{\frac{6e^{\gamma}}{\pi^2}\left[\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1\right]\right\}, \tag{1.2}$$

for every $x \ge c$, where $c = 1.461\dots$ is the unique positive zero of the digamma function ψ and $\gamma = 0.577\dots$ is the Euler–Mascheroni constant. In 2010, Mortici [28] discovered the asymptotic formula

$$\Gamma(x) \sim \frac{\sqrt{2\pi}}{e}\exp\left[\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1\right], \quad x \to \infty.$$

In 2011, the double inequality (1.2) was generalized in [11, Theorem 2] to a monotonicity property which reads that the function

$$f_{s,t}(x) = \begin{cases} \dfrac{g_{s,t}(x)}{[g'_{s,t}(x) - 1]\exp[g'_{s,t}(x)] + 1}, & x \ne c \\ \dfrac{1}{g''_{s,t}(c)}, & x = c \end{cases}$$

is decreasing for $|t-s| < 1$ and increasing for $|t-s| > 1$ in $x \in (-\alpha, \infty)$, where s and t are real numbers, $\alpha = \min\{s,t\}$, $c \in (-\alpha, \infty)$, and

$$g_{s,t}(x) = \begin{cases} \dfrac{1}{t-s}\displaystyle\int_c^x \ln\left[\frac{\Gamma(u+t)}{\Gamma(u+s)}\frac{\Gamma(c+s)}{\Gamma(c+t)}\right] d\,u, & s \ne t \\ \displaystyle\int_c^x [\psi(u+s) - \psi(c+s)]\, d\,u, & s = t \end{cases}$$

for $x \in (-\alpha, \infty)$.

In 2000, Elezović et al. [8] found the single-sided inequality

$$\psi'(x) < e^{-\psi(x)}, \quad x > 0. \tag{1.3}$$

This inequality is closely related to the monotonicity and convexity of the function

$$\mathcal{Q}(x) = e^{\psi(x+1)} - x$$

on $(-1, \infty)$. See also [13] and plenty of references therein. By the way, as a conjecture posed in [16, Remark 3.6], the complete monotonicity of the function $\mathcal{Q}(x)$ on $(0, \infty)$ still keeps open. An infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^k f^{(k)}(x) \ge 0$ on I for all $k \ge 0$. This class of functions has applications in approximation theory, asymptotic analysis, probability, integral transforms, and the like. See [7, Chapter 14], [25, Chapter XIII], [60, Chapter 1], and [61, Chapter IV]. For more information on the functions

$$\mathcal{Q}'(x-1) = \psi'(x)e^{\psi(x)} - 1 \quad \text{and} \quad \mathcal{Q}''(x-1) = \left\{\psi''(x) + [\psi'(x)]^2\right\}e^{\psi(x)},$$

please see the papers [14–16,18–20,35,37,40–43,46–50,53–55,63,64], the expository and survey articles [38,39,51,52], and a number of references cited therein. In 2011, Batir [5, Theorem 2.7] obtained the double inequality

$$(x+a^*)e^{-2\psi(x+1)} < \psi'(x+1) \le (x+b^*)e^{-2\psi(x+1)}, \quad x > 0, \tag{1.4}$$

where the constants $a^* = \frac{1}{2}$ and $b^* = \frac{\pi^2}{6e^{2\gamma}} = 0.518\dots$ are the best possible. In other words, Batir [5] proposed the approximation formula

$$\psi'(x+1) \sim (x+a)e^{-2\psi(x+1)}, \tag{1.5}$$

where a is a constant. A numerical computation shows that the approximation (1.5) gives a better result when choosing $a = a^*$, rather than the value $a = b^*$. This fact is somewhat expected as Batir obtained (1.4) as a result of the decreasing monotonicity of the function

$$\theta(x) = \psi'(x+1)e^{2\psi(x+1)} - x$$

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