Applied Mathematics and Computation 271 (2015) 567–580

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stability and convergence of radial basis function finite difference method for the numerical solution of the reaction–diffusion equations

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ARTICLE INFO

Keywords:
Radial basis function
Finite difference
Reaction–diffusion equation
Generalized multiquadric (GMQ)
Optimal shape parameter

ABSTRACT

Stability, convergence and application of radial basis function finite difference (RBF-FD) scheme is studied for solving the reaction–diffusion equations (RDEs). We show that the explicit RBF-FD method is stable, and stability condition depends on the shape parameter of related radial basis function.The generalized multiquadric (GMQ) is applied as radial basis function and weight coefficients are explicitly presented for equispaced node distribution. Also, two methods are presented to compute the optimal shape parameter. The combination of these methods with the GMQ-FD method will produce two efficient algorithms for numerical solution of RDEs: the variable GMQ-FD (VGMQ-FD) and the constant GMQ-FD (CGMQ-FD). We test the scheme on traveling wave and compare its accuracy with the conventional finite difference method (FDM).

1. Introduction

Mairhuber–Curtis showed that to reconstruct multivariate function from scattered data at N distinct nodes $\{x_1, x_2, \ldots, x_N\}$ in domain $\Omega \subseteq \mathbb{R}^s$ $(s \geq 2)$, there exist no Haar spaces of continuous functions [16]. For example, if we choose s-variate polynomials space of degree less than k ($\prod_k (\mathbb{R}^s)$) then existence and uniqueness of interpolation polynomial depends on both space dimension ($N \leq dim(\prod_k (\mathbb{R}^s))$) and arrangement of nodes in domain [12]. The latter leads to the special meshes generation on domain Ω which is still a time consuming process, arduous and fraught with pitfalls. Hence, application of the classical polynomial-based methods (such as FDM, FEM and FVM), which are also listed in the category of mesh-based methods, has faced with restrictions for numerical solution of PDEs.

To get rid of the above problems, the Mairhuber–Curtis theorem suggests that the interpolation space should be node dependent. Radial basis functions (RBFs) provide this condition [16]. RBFs are conditionally positive definite of order $m \geq 0$ which all their linear combinations with the possible addition of a polynomial term generate an interpolation space. This space guarantees existence of unique interpolation function for arbitrary data at N distinct nodes. Therefore, a RBF facilitates the evaluation of the interpolant without using a mesh. In addition to the approximation theory, RBFs are growing in popularity for solving partial differential equations [14,20–22,36,38]. In some of RBFs-based methods, the accuracy of method generally increases for flat RBF. This is very unfortunate because in the case of flat, the interpolation matrix is very ill-conditioned. But the numerical results illustrate this disadvantage depends on both underlying PDE and the associated method. That is, there may exist a RBF-based

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http://dx.doi.org/10.1016/j.amc.2015.09.034
0096-3003/

method or a PDE such that either not sensitive to the shape parameter or optimal shape parameter occurs in small values. Among the methods we can mention to RBF-FD method, which combines the formulation simplicity of FDM and the meshfree property of RBF. This method was proposed by Shu et al. [37]. Also, Wright [41] and Tolstykh and Shirobokov [39] introduced this method independently and approximately at the same time to solve the ODEs and elasticity problems, respectively. For flat RBF, Wright and Fornberg [42] showed that RBF-FD method is equivalent to the conventional finite difference scheme. Bayona et al. [6,7] proposed two methods to compute the optimal shape parameter of MQ-FD method for solving ordinary and time-independent partial differential equations. In some literature the RBF-FD method is used with other names such as *local radial basis function method* [35] and *local radial basis functions based differential quadrature collocation method* [15].

In this work, we apply the RBF-FD method for numerical study reaction–diffusion equations. A reaction–diffusion equation comprises a reaction term and a diffusion term, i.e. the typical form is as follows:

$$u_t = u_{xx} + f(u), \tag{1}$$

where $u = u(x, t)$ is a state variable and describes density/concentration of a substance, a population, ...at position $x \in \Omega \subset \mathbb{R}$ and at time t. So the first term on the right hand side describes the "diffusion". The second term, $f(u)$, is a smooth function $f : \mathbb{R} \to \mathbb{R}$ and describes processes with really change the present u, i.e. something happens to it (birth, death, chemical reaction, ...) [24]. We assume the nonlinear term $f(u)$ satisfies the Lipschitz condition $|f(u_1) - f(u_2)| \le L|u_1 - u_2|$.

Such equations appear in many applications [25]: the choice $f(u) = u(1 - u)$ yields Fisher's equation that was originally used to describe the spreading of biological populations [18], the Newell–Whitehead–Segel equation with $f(u) = u(1 - u^2)$ to describe Rayleigh–Benard convection [33], the Nagumo equation with $f(u) = u(1 - u)(u - \alpha)$ and $0 < \alpha < 1$ has been obtained as one of the set of equations modeling the transmission of electrical pulses in a nerve axon [32], and $f(u) = u^2 - u^3$ yields Zeldovich equation that arises in the combustion theory [19].

Exact solutions for the reaction–diffusion equations were given by Feng et al. [17] using means of the Cole–Hopf transformation and the Lie symmetry method, Li and Guo [29] by the first integral method, Abbasbandy [1] using the homotopy analysis method and Ağrseven and Özis [2] by homotopy perturbation method. Also Kudryashov [26,27] presented two new methods for finding exact solutions of Eq. (1). However , as in many of the applications considered in [31], $f(u)$ is non-linear, then it is not usually possible to obtain general exact analytical traveling wave solutions. Hence, some numerical methods have been applied to give the approximate solution for the reaction–diffusion problems [30]. For numerical research on Eq. (1), we can refer to works of Al-Khaled [3] by the Sinc collocation method, Uddin [40] using the nodal integral scheme, Chen et al. [11] by non-standard finite difference methods, Dehghan and Fakhar-Izadi [13] using pseudospectral methods, Bastani and Salkuyeh [4] by a sixth-order compact finite difference (CFD6) scheme, Gorder and Vajravelu [23] by a variational technique, Bhrawy [9] by a Jacobi–Gauss–Lobatto collocation method, Li and Ding [28] by a higher order finite difference method and Ramos [34] using exponential methods.

The rest of this paper is divided into five sections. The GMQ-FD scheme is briefly reviewed in Section 2. Two algorithms VGMQ-FD and CGMQ-FD are developed for the RDEs in Section 3 and their convergence and stability conditions are obtained. Section 4 is devoted to the numerical results and discussion. Section 5 summarizes the paper.

2. GMQ-FD method and its weight coefficients

Given a point set $X = \{x_1, \dots, x_N\}$, derivatives of function $u(x)$ can be evaluated approximately using only function values at the nodes [8]:

$$u^{(m)}(x_i) \simeq \tilde{u}^{(m)}(x_i) = \sum_{i=1}^{n_i} w_{ij}^{(m)} u(x_j), \tag{2}$$

where n_i is the number of nodes in support domain of ith node.

The key point in above formulae is the obtaining weight coefficients $w_{ij}^{(m)}$. In classic finite difference method (FDM), the polynomials are used as test functions for this work.

In spite of low cast and easy performance for the nonlinear equations, due to requirement of mesh generation on the domain, the computational cost of FDM is relatively high for problems with complex geometry in the high dimensions [16].

To eliminate disadvantages of FDM, Shu et al. [37] used RBFs as test function instead of polynomials. Hence, the relation (2) changes to the following form:

$$\varphi^{(m)}(\|x_i - x_k\|) = \sum_{j=1}^{n_i} w_{i_j}^{(m)} \varphi_k(x_{i_j}); \qquad k = i_1, \dots, i_{n_i}, \tag{3}$$

where φ is a radial basis function and $\varphi_k(x_{i_j}) = \varphi(\|x_{i_j} - x_k\|)$. The common choices for RBFs are listed in Table 1. We can write the relation (3) in the matrix form as $\Psi_{n_i}^{(m)} = A_{n_i} w_{n_i}^{(m)}$, where

$$\Psi_{n_i}^{(m)} = \left[\varphi_{i_1}^{(m)}(x_i) \quad \varphi_{i_2}^{(m)}(x_i) \quad \dots \quad \varphi_{i_{n_i}}^{(m)}(x_i)\right]^T,$$