Applied Mathematics and Computation 271 (2015) 594–608

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the Quadratic Eigenvalue Complementarity Problem over a general convex cone

Carmo P. Brás [a,*], Masao Fukushima [b], Alfredo N. Iusem [c], Joaquim J. Júdice [d]

[a] Centro de Matemática e Aplicações (CMA) and FCT/UNL, Caparica 2829-516, Portugal
[b] Faculty of Science and Engineering, Nanzan University, Nagoya 466-8673, Japan
[c] Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil
[d] Instituto de Telecomunicações, Universidade de Coimbra-Polo II, Coimbra 3030-290, Portugal

ARTICLE INFO

MSC:
65F15
90C33
90C30
90C26

Keywords:
Eigenvalue problems
Complementarity problems
Nonlinear programming
Global optimization

ABSTRACT

The solution of the Conic Quadratic Eigenvalue Complementarity Problem (CQEiCP) is first investigated without assuming symmetry on the matrices defining the problem. A new sufficient condition for existence of solutions of CQEiCP is presented, extending to arbitrary pointed, closed and convex cones a condition known to hold when the cone is the nonnegative orthant. We also address the symmetric CQEiCP where all its defining matrices are symmetric. We show that, assuming that two of its defining matrices are positive definite, this symmetric CQEiCP reduces to the computation of a stationary point of an appropriate merit function on a convex set. Furthermore, we discuss the use of the so called Spectral Projected Gradient (SPG) algorithm for solving CQEiCP when the cone of interest is the second-order cone (SOCQEiCP). A new algorithm is designed for the computation of the projections required by the SPG method to deal with SOCQEiCP. Numerical results are included to illustrate the efficiency of the SPG method and the new projection technique in practice.

1. Introduction

Given matrices $B, C \in \mathbb{R}^{n\times n}$, the *Eigenvalue Complementarity Problem* (to be denoted EiCP(B, C), see e.g. [29,30]), consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda Bx - Cx, \tag{1}$$

$$w \geq 0, x \geq 0, \tag{2}$$

$$x^t w = 0, \tag{3}$$

$$e^t x = 1, \tag{4}$$

with $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$. The last normalization constraint has been introduced, without loss of generality, in order to prevent the x component of a solution to vanish. The matrix B is usually assumed to be positive definite (PD). The problem has many applications in engineering (see [1,27,30]), and can be seen as a generalization of the well-known generalized eigenvalue problem,

* Corresponding author. Tel.: +351 212948388.
E-mail addresses: mb@fct.unl.pt (C.P. Brás), fuku@nanzan-u.ac.jp (M. Fukushima), iusp@impa.br (A.N. Iusem), joaquim.judice@co.it.pt (J.J. Júdice).

http://dx.doi.org/10.1016/j.amc.2015.09.014

denoted GEiP (see e.g. [18]). Indeed, GEiP consists of solving (1) with $w = 0$, and a solution (λ, x) of GEiP is just an eigenvalue and eigenvector of the matrix $B^{-1}C$ in the usual sense, when B is nonsingular. If a triplet (λ, x, w) solves EiCP, then the scalar λ is called a *complementary eigenvalue* and x is a *complementary eigenvector* associated with λ for the pair (B, C). The condition $x^t w = 0$ and the nonnegative requirements on x and w imply that either $x_i = 0$ or $w_i = 0$ for $1 \leq i \leq n$. These two variables are called complementary.

It is easy to prove that under strict copositivity of B, EiCP(B, C) always has a solution, because it can be reformulated as the variational inequality problem VIP$(\bar{F}, \Omega)$ with feasible set $\Omega = \{x \in \mathbb{R}^n : e^t x = 1, x \geq 0\}$ and operator $\bar{F} : \Omega \to \mathbb{R}^n$ given by

$$\bar{F}(x) = \frac{x^t C x}{x^t B x} Bx - Cx, \tag{5}$$

see [22]. Note that $\bar{F}$ is continuous in Ω as a consequence of the strict copositivity of B, and that Ω is convex and compact. It is well known that these two conditions ensure existence of solutions of VIP$(\bar{F}, \Omega)$, see [11]. In particular this result holds when B is PD (see [22]).

A number of techniques have been proposed for solving the EiCP and its extensions, see e.g. [2,6,14,15,20–23,26,28,29,32,33].

Recently an extension of the EiCP has been introduced in [31], where some applications are highlighted. It has been named *Quadratic Eigenvalue Complementarity Problem* (QEiCP), and it differs from EiCP through the existence of an additional quadratic term on λ. Its formal definition follows.

Given $A, B, C \in \mathbb{R}^{n\times n}$, QEiCP$(A, B, C)$ consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{6}$$

$$w \geq 0, x \geq 0, \tag{7}$$

$$x^t w = 0, \tag{8}$$

$$e^t x = 1, \tag{9}$$

where, as before, $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$. As in the case of the EiCP, the normalization constraint (9) has been introduced, without loss of generality, for preventing the x component of a solution of the problem from vanishing. Note that QEiCP(A, B, C) reduces to EiCP$(B, -C)$ when $A = 0$. The λ component of a solution of QEiCP(A, B, C) is called a *quadratic complementary eigenvalue* for A, B, C, and the x component a *quadratic complementary eigenvector* for A, B, C associated with λ.

The case of the symmetric QEiCP, i.e., when A, B and C are symmetric matrices and $-C$ is the identity matrix, has been analyzed in [13], where each instance of QEiCP with $n \times n$ matrices is related to an instance of EiCP with $2n \times 2n$ matrices. A new approach for solving the nonsymmetric QEiCP by a similar reduction has been recently studied in [7].

In this paper, we study the Conic Quadratic Eigenvalue Complementarity Problem (CQEiCP). This problem has been introduced in [31] as an interesting extension of QEiCP. It is defined as follows.

Given $A, B, C \in \mathbb{R}^{n\times n}$, a closed, convex and pointed cone $\mathcal{K} \subset \mathbb{R}^n$ and a vector $a \in \text{int}(\mathcal{K}^*)$, CQEiCP$(A, B, C)$ consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{10}$$

$$x \in \mathcal{K}, \quad w \in \mathcal{K}^*, \tag{11}$$

$$x^t w = 0, \tag{12}$$

$$a^t x = 1. \tag{13}$$

As before, the normalization constraint (13) prevents $x = 0$ from being a solution of the problem. When $A = 0$, i.e., when the first term in the right hand side of (10) is absent, CQEiCP becomes the so called Conic Eigenvalue Complementarity Problem. This problem is denoted by CEiCP(B, R) and is defined by the constraints (11)–(13) and

$$w = \lambda Bx - Rx,$$

which replaces (10). Hence CEiCP(B, R) = CQEiCP$(0, B, -R)$. It is known (see [32]), that CEiCP(B, R) has a solution whenever $\mathcal{K}$ is closed, convex and pointed and B is a PD matrix. CQEiCP may lack solutions even when the leading matrix A is PD. Indeed, if we consider CQEiCP$(I, 0, I)$ with an arbitrary cone $\mathcal{K}$, then premultiplying (10) by x and using (12), one gets $0 = (\lambda^2 + 1)\|x\|^2$, which has no solution $\lambda \in \mathbb{R}$ and $x \neq 0$. This difference between CEiCP and CQEiCP in terms of existence of solutions mirrors the elementary fact that linear equations in one real variable always have solutions, while quadratic equations may fail to have them.

Thus, the issue of conditions on (A, B, C) ensuring existence of solutions of CQEiCP(A, B, C) is a relevant one. In [31], the concepts of co-regularity and co-hyperbolicity of (A, B, C) were introduced, ensuring existence of solutions of CQEiCP(A, B, C). For the case of QEiCP (i.e., when $\mathcal{K} = \mathbb{R}^n_+$), it has been shown in [7] that existence of solutions of QEiCP is also guaranteed when the matrix A is strictly copositive and the matrix $-C$ is not an S_0 matrix. In order to establish this result, QEiCP is transformed into a $2n$-dimensional EiCP problem by using an auxiliary vector $y \in \mathbb{R}^n$ such that $y = \lambda x$.

In this paper we propose a new transformation of CQEiCP into CEiCP (for a general closed, convex and pointed cone $\mathcal{K}$) that differs from the one introduced in [7] by the introduction of a PD matrix E. Using this transformation, we will establish in Section 2 the existence of solutions of CQEiCP under hypotheses different from those demanded in [31].

In Section 3, we show that the solution of the symmetric CQEiCP (i.e., when the matrices A, B and C are symmetric), assuming that both A and $-C$ are PD matrices, reduces to the computation of a stationary point of a special fractional quadratic function