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On an exact penalty function method for nonlinear mixed discrete programming problems and its applications in search engine advertising problems

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ABSTRACT

In this paper, we study a new exact and smooth penalty function for the nonlinear mixed discrete programming problem by augumenting only one variable no matter how many constraints. Through the smooth and exact penalty function, we can transform the nonlinear mixed discrete programming problem into an unconstrained optimization model. We demonstrate that under mild conditions, when the penalty parameter is sufficiently large, optimizers of this penalty function are precisely the optimizers of the nonlinear mixed discrete programming problem. Alternatively, under some mild assumptions, the local exactness property is also presented. The numerical results demonstrate that the new penalty function is an effective and promising approach. As important applications, we solve an increasingly popular search engine advertising problem via the new proposed penalty function.

1. Introduction

Many practical optimum design problems in the engineering and finance management fields can be formulated as nonlinear mixed discrete programming problems. For example, in the stocks trade marketing, a stock holder is merely requested to buy or sell integer number stocks with others. Thus, in order to maximize stock holders' total revenues, a mixed discrete programming problem should be considered.

Some popular methodologies for nonlinear mixed discrete programming problems are designed, such as the cutting plane method [20] and the Branch-and-Bound method (BBM) [27]. In both of the methods, a nonlinear mixed discrete programming problem is solved through some relaxation method, for example, ignoring any discrete restrictions. However, these two methods, as the case when applied in integer programming, are far from efficient for nonlinear mixed discrete programming problems. The most prominent difficulty is that solving such problems may take much more time as the number of discrete variables increases. In particular, the cutting plane method, as we know, needs to generate a great number of cuts, as well as for the Branch-and-Bound method, if the number of discrete variables is large, the branching process is very expensive numerically. The main idea of another typical class of available approaches for nonlinear mixed discrete programming problems is to transform discrete formulations into continuous ones. Nowadays, continuous formulations of discrete problems have been widely developed in

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the literature, for example, [6,12,19,25,26,29,35,36] and references therein for relevant results on the equivalence between integer and continuous programming; [1,3,10,11,31,32,34] and references therein for some relevant applications where continuous formulations have been successfully used. The main idea of most literature is to design continuous formulations that enable optimization methods for continuous problems to be applied to certain types of discrete optimization problems.

It is well known that a penalty function algorithm is an important method for solving constrained optimization problems, which, by augmenting a penalty term, transforms the original problem into a single or a sequence of unconstrained problems. For classical penalty function methods, we need to make the penalty parameter infinitely large in a limiting sense to recover an optimal solution of the original problem, which causes some numerical instability in implementation. To avoid this difficulty, we require the penalty function to satisfy the following property: recovering an exact solution of the original problem for reasonable finite values of the penalty parameter. The penalty function possessing this property is known as an exact penalty function. However, it should be mentioned that some exact penalty functions have a disadvantage that it either needs Jacobian [14,18,22,28,29] or is no longer smooth (l_1 or l_∞ penalty functions etc. [2,8,9,13,15,17,23,30,38,42]).

Recently, a new exact penalty function is presented by Huyer and Neumaier [24] for an equality constrained minimization problem. The exact penalty property of the corresponding penalty function is discussed thoroughly in [24], e.g., for smooth case, the exact penalty property is established if the equality constrained function is level-bounded and every point in the level set satisfies Mangasarian–Fromovitz constraint qualification as in [24, Theorem 2.1]; for nonsmooth case, through standard smoothing approximation techniques, sufficient conditions are derived for the local exactness property (see [24, Theorem 5.3]).

The reason why the new penalty function has significant differences from classical simple exact penalty functions is that the associated penalty function in [24] has good smoothness property which is not shared by classical simple and exact penalty functions (see [4,21]). In addition, by the definition of classical penalty functions ([4]), the values of the penalty term are zero on the feasible set and positive outside the feasible set, which is not the case for the penalty term of the penalty function designed in [24].

In this paper, we are mainly concerned with the following nonlinear mixed discrete programming problem

$$\begin{cases} \min & f(x,y) \\ \text{s.t.} & h_l(x,y)=0 \quad \forall l=1,2,\dots,L_1, \\ & g_\ell(x,y)\le 0 \quad \forall \ell=1,2,\dots,L_2. \\ & x=[x_1,x_2,\dots,x_n]^\top\in\mathbb{R}^n, \\ & y=[y_1,y_2,\dots,y_m]^\top\in D_1\times D_2\cdots\times D_m, \end{cases} \tag{$\bar{\mathrm{P}}$}$$

where L_1 and L_2 are positive integers. The functions $h_l, l=1,2,\dots,L_1$ and $g_\ell, \ell=1,2,\dots,L_2$ are continuously differentiable with respect to all their arguments. For $i=1,2,\dots,m$, $D_i=\{a_{i,1},a_{i,2},\dots,a_{i,J_i}\}$ is a discrete point set with J_i elements, where $a_{i,j}, j=1,2,\dots,J_i$ are given discrete values.

Continuous formulations in [41] are employed to tackle with discrete variables. Each of the discrete variables is represented by a set of new binary variables $v_{i,j}, i=1,2,\dots,m, j=1,2,\dots,J_i$ with linear constraints. By introducing auxiliary quadratic functions, it is shown that the new employed binary variables become continuous on the interval [0, 1]. Consequently, the original mixed discrete programming problem is transformed into a continuous nonlinear programming problem. We find that, unlike others, the continuous formulations in [41] introduce new quadratic functions into constraints in order to enable discrete optimizations into continuous ones. Since these formulations are at most quadratic, and hence, they will not increase the number of local optima. Furthermore, compared to continuous formulations in [1,3,10,11,31,32,34], smoothness property is the most important feature of this developed continuous formulation. In consideration of the purpose of this paper, which is to design a new exact and smooth penalty function to tackle this transcribed nonlinear continuous optimization problem, continuous formulations in [41] are more suitable.

Motivated by the idea in [24], we propose a new exact and smooth penalty function based on the exact penalty function method introduced in [24] for nonlinear mixed discrete programming problems. The merit function is considered as a function of x and ε simultaneously which has good smoothness and exactness properties, without involving gradient and Jacobian matrices. We present the result that, if a local optimal solution of the penalty problem satisfies the linearly independent constraint qualification, then the minimizer has the expression of ($x^\star$, 0). In addition, we derive a quite useful conclusion that the minimizer ($x^\star$, $\varepsilon^\star$) of the penalty problem satisfies $\varepsilon^\star=0$ if and only if $x^\star$ solves the original problem. This property allows us to view the introduced variable ε an indicator variable for a local (global) minimizer of the original problem. Besides the above properties, we provide that the penalty problem possesses exactness property, that is, there exists a threshold, when penalty parameter is greater than this threshold, the minimizer of penalty problem is exactly the one of original problem. Furthermore, under mild conditions, the local exactness property is accurately characterized even if the objective and constraint functions are not necessarily smooth.

Subsequently, we apply the proposed penalty function to solve search engine advertising problems. As well known, internet search engines, such as Google and Yahoo! provide a service where after a user has searched a specific term, sponsored links may be displayed in the front page in addition to search results. Sponsored links offer advertisers a more targeted method of advertising than traditional forms of advertising such as TV commercials, because they are customized. Motivated by industrial practice, we model the search engine advertisement auction problem as an integer programming, which maximizes the search engine revenue by choosing the optimal advertisers' bidding positions. Undoubtedly, the larger dimensions of variables, the more computational challenges incur. Here, we utilize the proposed exact and smooth penalty function for the formulated integer