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A system of matrix equations with five variables

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ABSTRACT

In this paper, we give some necessary and sufficient conditions for the consistence of the system of quaternion matrix equations

$$\begin{aligned}
&A_1X = C_1,\ YB_1 = D_1,\\
&A_2W = C_2,\ ZB_2 = D_2,\\
&A_3V = C_3,\ VB_3 = C_4,\ A_4VB_4 = C_5,\\
&A_5X + YB_5 + C_6W + ZD_6 + E_6VF_6 = G_6,
\end{aligned}$$

and constitute an expression of the general solution to the system when it is solvable. The outcomes of this paper encompass some recognized results in the collected works. In addition, we establish an algorithm and a numerical example to illustrate the theory constructed in the paper.

1. Introduction

In this paper, the notation $\mathbb{H}$ interprets the quaternion algebra

$$\mathbb{H} = \{p_0 + p_1i + p_2j + p_3k \mid i^2 = j^2 = k^2 = ijk = -1,\ p_0, p_1, p_2, p_3 \in \mathbb{R}\}$$

where $\mathbb{R}$ is the real number filed, and $\mathbb{H}^{m\times n}$ for the set of all quaternion matrices of dimension $m \times n$ over $\mathbb{H}$. Zhang gave comprehensive study on quaternions and quaternion matrices in [51]. Quaternion's applications in signal processing, computer graphics, quantum physics and color imaging can be observed in [2,21,22,25,26,30,37–40]. I represents an identity matrix with conformable shape. For a quaternion matrix A, the notations A^* and $r(A)$ represent the conjugate transpose, and the rank of A, respectively. The Moore–Penrose inverse of A is denoted by $A\dagger$ and is defined to be the solution of the following matrix equations

$$AA^{\dagger}A = A, \quad A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad (AA^{\dagger})^* = AA^{\dagger}, \quad (A^{\dagger}A)^* = A^{\dagger}A.$$

In addition, $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$ are two projectors generated by A, respectively and these are idempotent and Hermitian by the definition of the Moore–Penrose inverse. Some findings on Moore–Penrose inverse can be consulted in [6] and [33].

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Linear matrix equations have been a crucial role in matrix theory and its applications. Many authors have adopted different techniques to determine the solutions of matrix equations (see, e.g. [7–11,13,17,18,23,28,29,36,41,42,44,46–48,53,55]). For instance, Liao and Bai [14] calculated the least-square solution of $AXB = C$ over symmetric positive semi-definite matrices X. Roth examined the necessary and sufficient condition for the consistency of

$$AX - YB = C$$

in [34]. Baksalary and Kala established the general solution to the consistent matrix equation

$$AXB + CYD = E \tag{1.1}$$

in [1]. Eq. (1.1) was also studied by different authors in [16,24,31]. The researchers in [15] examined the least-squares symmetric and skew-symmetric solutions of

$$AXA^T + BYB^T = C$$

with the least norm.

The general solution to

$$A = BX + (BX)^* \tag{1.2}$$

was examined by Braden [3] in 1998. The general solution to (1.2) in the framework of linear bounded operator on Hilbert spaces was calculated by Djordjević in [12]. The general solution to

$$A = DXE + (DXE)^* \tag{1.3}$$

was researched by Cao in [5]. Xu et al. also explored the operator Eq. (1.3) in [50].

Recently, Wang and He [45] found the expression of the general solution of the consistent quaternion matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1.$$

Zhang calculated the general solution to the system of matrix equations

$$\begin{aligned}
&A_1X_1 = C_1, \quad X_2B_1 = D_1,\\
&A_2X_3 = C_2, \quad X_3B_2 = D_2,\\
&A_3X_4 = C_3, \quad X_4B_3 = D_3,\\
&A_4X_1 + X_2B_4 + C_4X_3D_4 + C_5X_4D_5 = E_1,
\end{aligned}$$

in [52]. In 2013, Zhang and Wang [54] investigated the solvability conditions and the exact solution of the matrix equations

$$\begin{aligned}
&A_3ZB_3 = C_3,\\
&A_2Z = C_2, \quad ZB_2 = D_2,\\
&A_1X = C_1, \quad YB_1 = D_1,\\
&A_4X + YB_4 + C_4ZD_4 = E_1,
\end{aligned} \tag{1.4}$$

by using extremal ranks technique.

Observe that (1.4) is a particular case of the following system of quaternion matrix equations

$$\begin{cases}
A_1X = C_1, \quad YB_1 = D_1,\\
A_2W = C_2, \quad ZB_2 = D_2,\\
A_3V = C_3, \quad VB_3 = C_4, \quad A_4VB_4 = C_5,\\
A_5X + YB_5 + C_6W + ZD_6 + E_6VF_6 = G_6.
\end{cases} \tag{1.5}$$

To our best knowledge, there has been little information on the general solution of (1.5). Inspired by the intensive applications of linear matrix equations in control theory [19,20], stability analysis, model reduction, stability analysis and optimal control [32,35] and by keeping the interests and robust applications of quaternion matrices in mind, we in this paper give the solvability conditions and the expression of the general solution to the solvable system (1.5) over $\mathbb{H}$ by using different method from one in [54]. To gain our ambition, we use the rank equalities and generalized inverses technique to construct the general solution to the system (1.5) which is more easier and practical than the method used in [54]. Notice that the system (1.5) is not a simple modification of (1.4) because more equations mean more parameters and variables must be encountered.

The remainder of this paper is comprised as follows. In Section 2, we create some necessary and sufficient conditions for the existence of a solution to the system (1.5) and also constitute its general solution when it is solvable. In Section 3, we consider a particular case of (1.5). In Section 4, we establish an algorithm and a numerical example to interpret our key result of this paper. In Section 5, we give a conclusion to this paper.