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Characteristics polynomial of normalized Laplacian for trees

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ABSTRACT

Here, we find the characteristics polynomial of normalized Laplacian of a tree. The coefficients of this polynomial are expressed by the higher order general Randić indices for matching, whose values depend on the structure of the tree. We also find the expression of these indices for starlike tree and a double-starlike tree, $H_m(p, q)$. Moreover, we show that two cospectral $H_m(p, q)$ of the same diameter are isomorphic.

1. Introduction

Let $\Gamma = (V, E)$ be a simple finite undirected graph of order n. Two vertices $u, v \in V$ are called neighbors, $u \sim v$, if they are connected by an edge in E, $u \nsim v$ otherwise. Let d_v be the degree of a vertex $v \in V$, that is, the number of neighbors of v. The *normalized Laplacian matrix* [7], $\mathcal{L}$, of Γ is defined as:

$$\mathcal{L}(\Gamma)_{u,v} = \begin{cases} 1 \text{ if } u = v \text{ and } d_v \neq 0, \\ -\frac{1}{\sqrt{d_u d_v}} \text{ if } u \sim v, \\ 0 \text{ otherwise.} \end{cases} \quad (1)$$

This $\mathcal{L}$ is similar to the normalized Laplacian Δ defined in [2,20]. Let $\phi_\Gamma(x) = det(xI - \mathcal{L})$ be the characteristics polynomial of $\mathcal{L}(\Gamma)$. Let us consider $\phi_\Gamma(x) = a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n$. Now if Γ has no isolated vertices then $a_0 = 1, a_1 = n, a_2 = \frac{n(n-1)}{2} - \sum_{i\sim j} \frac{1}{d_i d_j}$ and $a_n = 0$ (for some properties of $\phi_\Gamma(x)$ see [8]). The zeros of $\phi_\Gamma(x)$ are the eigenvalues of $\mathcal{L}$ and we order them as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. Γ is connected *iff* $\lambda_{n-1} > 0$. $\lambda_1 \leq 2$, the equality holds *iff* Γ has a bipartite component. Moreover, Γ is bipartite *iff* for each λ_i, the value $2 - \lambda_i$ is also an eigenvalue of Γ. See [7] for more properties of the normalized Laplacian eigenvalues.

1.1. General Randić index for matching

There are different topological indices. The degree based topological indices [11,21], which include *Randić index* [23], reciprocal *Randić index* [16], general *Randić index* [4], higher order *connectivity index* [3,22,24], *connective eccentricity index* [29,30], *Zagreb index* [1,9,12–14,18,27,28] are more popular than others.

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For any real number α, the general Randić index of a graph Γ is defined by Bollobás and Erdös (see [4]) as:

$$R_\alpha(\Gamma) = \sum_{i\sim j}(d_i d_j)^\alpha, \tag{2}$$

which is the general expression of the Randić index (also known as connectivity index) introduced by Randić in 1975 [23] by choosing $\alpha = -1/2$ in (2). For more properties of the general Randić index of graphs, we refer to [5,16,17,24–26].

The Zagreb index of a graph was first introduced by Gutman and Trinajstić [12] in 1972. For a graph Γ the first and the second Zagreb indices are defined by

$$Z_1(\Gamma) = \sum_{i\in V} d_i^2 \text{ and } Z_2(\Gamma) = \sum_{i\sim j} d_i d_j,$$

respectively. Now, for any positive integer p, we define the pth order general *Randić index for matching* as

$$R_\alpha^{(p)}(\Gamma) = \sum_{M_p\in\mathcal{M}_p(\Gamma)} \prod_{e\in M_p} s(e)^\alpha, \tag{3}$$

where $s(e) = d_u d_v$ is the *strength* of the edge $e = uv \in E$, M_p is a p-matching, that is, a set of p non-adjacent edges and $\mathcal{M}_p(\Gamma)$ is the set of all p-matchings in Γ. The first order general Randić index for matching with $\alpha = 1$ is the second Zagreb index, that is, $Z_2(\Gamma) = R_1^{(1)}(\Gamma)$. We take $R_\alpha(\Gamma) = R_\alpha^{(1)}(\Gamma)$ and

$$R_\alpha^{(0)}(\Gamma) = \begin{cases} 0 \text{ if } \Gamma \text{ is the null graph,} \\ 1 \text{ otherwise.} \end{cases}$$

If Γ is r-regular, then $R_\alpha^{(i)} = r^{2i\alpha} \mid \mathcal{M}_i(\Gamma) \mid$. The $R_{-1}^{(2)}$ for some known graphs are as follows: $R_{-1}^{(2)}(S_n) = 0, R_{-1}^{(2)}(P_n) = \frac{n^2-n-4}{32}, R_{-1}^{(2)}(C_n) = \frac{n(n-3)}{32}, R_{-1}^{(2)}(K_{p,q}) = \frac{(p-1)(q-1)}{4pq}$, and $R_{-1}^{(2)}(K_n) = \frac{3\binom{n}{4}}{(n-1)^4}$, where the notations, S_n, P_n, C_n, K_n and $K_{p,q}$ have their usual meanings.

Theorem 1.1. *For any real number α,*

$$0 \le R_\alpha^{(2)}(\Gamma) \le \frac{1}{2}\Big[R_\alpha(\Gamma)\Big]^2 - \frac{1}{2}R_{2\alpha}(\Gamma)$$

Proof.

$$\begin{aligned}[R_\alpha(\Gamma)]^2 &= \left[\sum_{e\in E}(s(e))^\alpha\right]^2 \\ &= \sum_{e\in E}(s(e))^{2\alpha} + 2\sum_{e_1,e_2\in\mathcal{M}_2(\Gamma)}(s(e_1)s(e_2))^\alpha + 2\sum_{e_1,e_2\notin\mathcal{M}_2(\Gamma)}(s(e_1)s(e_2))^\alpha\end{aligned}$$

which proves our required result. □

Clearly, for any two graphs Γ_1, Γ_2, and $p \ge 0$, $R_\alpha^{(p)}(\Gamma_1 \cup \Gamma_2) \ge R_\alpha^{(p)}(\Gamma_1) + R_\alpha^{(p)}(\Gamma_2)$. The equality holds, when $p = 1$ or one of the graphs is null.

It has been seen that the matching plays a role in the spectrum of a tree. In [6], some results on normalized Laplacian spectrum for trees have been discussed. Now, we derive (or express the coefficients of) the characteristics polynomial $\phi_T(x)$ of a tree T in terms of $R_{-1}^{(i)}(T)$.

2. The characteristics polynomial of normalized Laplacian for a tree

Theorem 2.1. *Let T be a tree with n vertices and maximum matching number k, then*

$$\phi_T(x) = \sum_{i=0}^{k}(-1)^i(x-1)^{n-2i}R_{-1}^{(i)}(T) \tag{4}$$

and the coefficients of ϕ_T are given by

$$a_p = \sum_{i=0}^{k}(-1)^i\binom{n-2i}{p-2i}R_{-1}^{(i)}(T). \tag{5}$$