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On boundary value problems for impulsive fractional differential equations

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ABSTRACT

In the present paper, we deal with the existence of solutions for the following impulsive fractional boundary value problem:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = f(t, u(t)), & t \neq t_j, \quad \text{a.e. } t \in [0, T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, n, \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (1/2, 1]$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, n$, are continuous functions, $a \in C([0, T])$ and

$$\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^-),$$

$${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+) = \lim_{t \to t_j^+}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t)),$$

$${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^-) = \lim_{t \to t_j^-}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t)).$$

By using critical point theory and variational methods, we give some new criteria to guarantee that the above-mentioned impulsive problems have at least one solution or infinitely many solutions. Some examples are also provided.

1. Introduction

The aim of this paper is to establish some results on the existence of solutions to the following class of boundary value problems for impulsive fractional differential equations:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = f(t, u(t)), & t \neq t_j, \quad \text{a.e. } t \in [0, T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, n, \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

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where the order of the derivative is $\alpha \in (1/2, 1]$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$ is a fixed (finite) sequence of impulse instants, the functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, n$, are continuous, $a \in C([0, T])$ and the following notation is used:

$$\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-),$$

$${}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+) = \lim_{t \to t_j^+} ({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t)),$$

$${}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-) = \lim_{t \to t_j^-} ({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t)).$$

Fractional differential equations have been an area of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [1–3] and the references therein.

In fact, the adequacy of fractional derivatives to describe the memory effects and hereditary properties in a great variety of processes makes fractional differential models interesting and with a great potential in applications, which is supported by the good adjustment between simulations and experimental data. As indicated in [4], the dynamics of natural systems are in many occasions complex, so that classical models might not be adequate. In this reference, the authors also show that the order of the fractional derivative is important to control the speed in which the trajectories of fractional systems move with respect to the critical point. This behavior is found by the authors of [4] as an interesting issue in relation with anomalous behavior appearing among competing species or in the study of diseases and justifies the applicability of fractional models in biology.

The authors of [5,6] consider some fractional models related to diffusion processes, in particular, fractional calculus is shown to be appropriate to simulate heat dissipation, based in the analogy existing between heat and electrical conduction in terms of a 0.5-order derivative. Reference [6] is a pioneer in the use of fractional models of diffusion processes together with lumped RC networks. A numerical study is made in [7] for the nonlinear time-fractional gas dynamics equation. On the other hand, in [8], it is proposed an impulsive nonlinear differential equation with fractional derivative with interesting applications to pest management and, besides, some contributions to the study of option price governed by a Black–Scholes equation with a time-fractional derivative can be found in [9].

In [10], the author considers the porous medium equation with a time-fractional derivative as a good model that can allow the presence of waiting-times in the medium, obtaining a very accurate solution and providing some useful explicit approximating formulas.

Some other recent applications that show the actual importance of fractional differential models are those about fractional damped dynamical systems [11], time-fractional Schrödinger equation [12], fractional hydrodynamical equations with applications on soft micromagnetic materials [13], fractional wave equations [14], time-fractional heat conduction problems [15–17], pressure-transient behavior of a well in a heterogeneous reservoir [18], Schrödinger equation with fractional Laplacian [19], fractional Euler–Lagrange equation modeling a fractional oscillator [20], feedback control systems on networks [21], image denoising [22], iterative learning control [23], neutron fractional diffusion equation which considers spatial-fractional derivatives and is useful to model anomalous diffusion observed in reactors due to production and absorption [24], or space fractional advection diffusion [25,26], among others. See also [27].

Concerning the use of critical point theory for the study of fractional differential equations, we mention some recent works. For instance, in [28–32], the existence of solutions for a class of fractional boundary value problems is considered and, in [33], critical point results are used to study the existence and multiplicity of solutions for a fractional boundary value problem with superquadratic, asymptotically quadratic or subquadratic nonlinearity.

On the other hand, the existence of infinitely many solutions to a class of fractional boundary value problems with nonsmooth potential is studied in [34] and an application of minmax methods in critical point theory for fractional boundary value problems can be found in [35]. Some contributions to fractional differential inclusions by using nonsmooth critical point theory are given in [36,37].

In [38], it is proved the existence and multiplicity of weak solutions for a class of fractional boundary value problems by using a critical point result included in [39]. The fractional Schrödinger equation is considered in [40] and, by applying variant Fountain theorems, the author proves the existence of infinitely many nontrivial solutions with high or small energy. The existence of at least three distinct weak solutions for a coupled system of nonlinear fractional differential equations is studied in [41] also from the point of view of variational approach. The Mountain pass theorem and Ekeland's variational principle are the tools used in [42] for the analysis of fractional Schrödinger equations. Furthermore, the multiplicity of solutions to a certain class of fractional Hamiltonian systems is considered in [43], while the existence of infinitely many solutions to this type of systems is studied in [44,45].

Differential equations with impulsive effects arise from many phenomena in the real world and describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background concerning the basic theory and some applications of impulsive differential equations, we refer the interested readers to [46–51]. There have been many different approaches to the study of the existence of solutions to impulsive fractional differential equations, such as topological degree theory, fixed point theory, upper and lower solutions method, monotone iterative technique and so on (see, for example, [52–58] and the references therein).