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# Quantitative convergence results for a family of hybrid operators

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ABSTRACT

We introduce a one parameter family of hybrid operators and study quantitative convergence theorems for these operators e.g. local and weighted approximation results and simultaneous approximation of derivatives. Further, we discuss the statistical convergence of these operators. Lastly, we show the rate of convergence of these operators to a certain function by illustrative graphics in Matlab.



## 1. Introduction

In order to generalize the Baskakov operators, Mihesan [17] proposed the following operators based on a non-negative constant $a$, independent of $n$ as

$$B_n^a(f;x) = \sum_{k=0}^{\infty} b_{n,k}^a(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$b_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{\sum_{i=0}^{k}\binom{k}{i}(n)_i a^{k-i}}{k!} \frac{x^k}{(1+x)^{n+k}},$$

and the rising factorial is given by $(n)_i = n(n+1)\ldots(n+i-1),\ (n)_0 = 1$. It was seen in [17] that $\sum_{k=0}^{\infty} b_{n,k}^a(x) = 1$. Obviously, if $a = 0$, we obtain at once the Baskakov basis function

$$b_{n,k}^0(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

By considering the generalized Baskakov basis functions, Erencin [5] proposed the Durrmeyer type operators, which for $a = 0$ reduce to the modified Baskakov type operators considered in [11]. In the literature several other forms of Baskakov–Durrmeyer operators have been discussed cf. [12–14]. Gonska and Păltănea [9] considered a class of one parameter operators of Bernstein–Durrmeyer type that preserve linear functions and constitute a link between the well known operators of Bernstein and their

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genuine Bernstein–Durrmeyer variants and established the quantitative Voronovskaja type theorems in terms of the first order and second order modulus of smoothness which generalize and improve the earlier results for these operators. In [10], the authors obtained the estimates for simultaneous approximation of derivatives for the one parameter family of operators studied in [9]. Păltănea [20] proposed the generalization of the well known Phillips operators by taking the generalized basis functions under integration based on certain parameter $\rho > 0$. Here, we propose a new kind of hybrid operators by considering the two generalized basis functions of [5,20].

For $\gamma > 0$, we define $C_\gamma[0, \infty) := \{f \in C[0, \infty); |f(t)| \le M_f e^{\gamma t}, \text{for some } M_f > 0\}$ endowed with the norm $|f|_\gamma = \sup_{t\in[0,\infty)} |f(t)|e^{-\gamma t}$, we propose the hybrid operators depending on two parameters $a$ and $\rho$ as follows:

$$L_n^{a,\rho}(f;x) = \sum_{k=1}^{\infty} b_{n,k}^a(x)\int_0^\infty s_{n,k}^\rho(t)f(t)dt + b_{n,0}^a(x)f(0), x \in [0,\infty) \tag{1.2}$$

where

$$s_{n,k}^\rho(t) = n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}.$$

It is observed that the operators (1.2) preserve only the constant functions.

**Special cases:**

(1) For $a = 0$ and $\rho = 1$, these operators include the well known operators introduced in [2].
(2) For $a = 0$ and $\rho \to \infty$, these operators reduce to the well known Baskakov operators.
(3) For $a > 0$ and $\rho \to \infty$, these operators reduce to the generalized Baskakov operators [17].

The aim of the present paper is to study some direct results in terms of the modulus of continuity of second order, the weighted space and the degree of approximation of $f^{(r)}$ by $L_n^{a,\rho(r)}(f;.)$. We also study the statistical convergence. The rate of convergence of the operators $L_n^{a,\rho}$ to a certain function is also illustrated through graphics in Matlab.

In what follows, let us assume that $0 < c < d < \infty$, $I = [c, d]$; $0 < c_1 < c_2 < d_2 < d_1 < \infty$ and $I_i = [c_i, d_i]$, $i = 1, 2$.

## 2. Basic results

For $m \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$, the $m$th order central moment of the generalized Baskakov operators $B_n^a$ is defined as

$$u_{n,m}^a(x) = B_n^a((t-x)^m;x) = \sum_{k=0}^{\infty} b_{n,k}^a(x)\left(\frac{k}{n}-x\right)^m.$$

**Lemma 1.** [1] *For the function $u_{n,m}^a(x)$, we have*

$$u_{n,0}^a(x) = 1,\quad u_{n,1}^a(x) = \frac{ax}{n(1+x)}$$

*and*

$$x(1+x)^2(u_{n,m}^a(x))' = n(1+x)u_{n,m+1}^a(x) - axu_{n,m}^a(x) - mx(1+x)^2u_{n,m-1}^a(x),\quad \text{for}\ \ m \ge 1. \tag{2.1}$$

*Consequently,*

*(i) $u_{n,m}^a(x)$ is a rational function of $x$ depending on the parameter $a$;*

*(ii) for each $x \in (0,\infty)$ and $m \in \mathbb{N}^0$, $u_{n,m}^a(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integer part of $\alpha$.*

**Lemma 2.** [1] *For each $x \in (0,\infty)$ and $r \in \mathbb{N}^0$, there exist polynomials $q_{i,j,r}(x)$ in $x$ independent of $n$ and $k$ such that*

$$\frac{d^r}{dx^r}b_{n,k}^a(x) = b_{n,k}^a(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j\frac{q_{i,j,r}(x)}{(p(x))^r},$$

*where $p(x) = x(1+x)^2$.*

**Lemma 3.** *For $m \in \mathbb{N}^0$, the mth order moment for the operators (1.2) defined as*

$$\mu_{n,m}^{a,\rho}(x) := L_n^{a,\rho}(t^m;x) = \sum_{k=1}^{\infty} b_{n,k}^a(x)\int_0^\infty s_{n,k}^\rho(t)t^m dt,$$

*we have $\mu_{n,0}^{a,\rho}(x) = 1$ and there holds the following recurrence relation:*

$$n(1+x)\mu_{n,m+1}^{a,\rho}(x) = x(1+x)^2(\mu_{n,m}^{a,\rho}(x))' + \left(nx(1+x) + ax + \frac{m(1+x)}{\rho}\right)\mu_{n,m}^{a,\rho}(x).$$