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Semigroup structural form for Bernstein-type operators and its applications

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A B S T R A C T

In this note, we provide a semigroup viewpoint for Bernstein-type operators. A main ingredient of our approach is to obtain some explicit semigroup representation formulae, which, as simple forms, are proposed for the fist time. As direct applications, we improve the corresponding results of De La Cal and Luquin (1992; 1994). Moreover, some kinds of Voronovskaja-type results are made by using general semigroup expansion. In the end, we discuss corresponding generalizations of integral operators with some examples of hybrid Durrmeyer type operators.

1. Introduction

Since the pioneering work of Hille and Philips ([16,18]), operator semigroups methods have been used with success in dealing with problems of Poisson approximation, and of convergence of Bernstein-type operators, which are exhibited as limits, whenever the parameters are conveniently chosen, of other ones ([7,8,17]). In preceding papers ([19–22]) the author provided explicit estimations for the rate of convergence of basically all known representation formulae for operator semigroups which have been shown to arise from a single general probabilistic representation theorem.

In this paper, we shall obtain some extra results unfinished in ([7,8,17]) through the establishment of representation formulae for Bernstein-type operators by bounded linear operators and (C_0)-semigroups, of which the present paper is an extended part.

In the following we list the operators involved in the mentioned results and introduce related operator semigroups, for full details one can refer to ([3,6,16,18–22]).

Let $C_B[0,\infty)$ be the space of continuous and bounded function on unbounded interval $[0,\infty)$, and let $f\in C_B[0,\infty)$, the BBH, Baskakov, Szász, and Gamma operators are defined respectively as follows:

$$L_n(f(u),x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) p_{n,k}\left(\frac{x}{1+x}\right). \tag{1.1}$$

$$M_n(f(u),x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) b_{n,k}(x). \tag{1.2}$$

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$$S_n(f(u),x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\pi_{n,k}(x). \tag{1.3}$$

$$G_n(f(u),x) := \frac{x^{-n}}{\Gamma(n)}\int_0^{\infty} f\left(\frac{u}{n}\right)u^{n-1}e^{-u/x}du. \tag{1.4}$$

Finally, for $f \in C[0,1]$ (the space of continuous function), Bernstein operators are defined by

$$B_n(f(u),x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x). \tag{1.5}$$

Where

$$b_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad \pi_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \quad p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}(k=0,1,2,\ldots).$$

In [7], De La cal and Luquin established the following theorem:

Theorem A. *Let $m \in \mathbb{N}^+$, $f \in C_B[0,\infty)$, then*

(a) For $n \ge x \ge 0$,

$$\left|B_{mn}\left(f(nu),\frac{x}{n}\right) - S_m(f(u),x)\right| \le \frac{2x}{n}\|f\|\min\{2,mx\}. \tag{1.6}$$

(b) For $x \ge 0$, $n = 1,2,\ldots,$

$$\left|L_{mn}\left(f\left(\frac{nu}{1+u}\right),\frac{x}{n}\right) - S_m(f(u),x)\right| \le 2\omega\left(f,\frac{x}{mn}\right) + \frac{x(1+x)}{n}2m||f||(1+2mx)e^{2mx}. \tag{1.7}$$

Among them, $\|f\| = \sup_{x\ge 0}|f(x)|$, $\omega(f,\delta) = \sup_{|x_1-x_2|\le\delta}|f(x_1)-f(x_2)|$.

In the later paper [8], the authors established the following Theorem:

Theorem B. *Let $m \in \mathbb{N}^+$, $f \in C_B[0,\infty)$, then*

$$(c)\left|M_{mn}\left(f(nu),\frac{x}{n}\right) - S_m(f(u),x)\right| \le \min\left\{\frac{2x}{n}||f||,\frac{mx^2}{2n}\omega_2\left(f,\frac{1}{m}\right)\right\}. \tag{1.8}$$

$$d)\left|M_{mn}\left(f\left(\frac{u}{n}\right),nx\right) - G_m(f(u),x)\right| \le \left(1+\frac{\Gamma(m+1/2)}{\Gamma(m+1)}\sqrt{x}\right)\omega\left(f,\frac{1}{\sqrt{n}}\right). \tag{1.9}$$

(e) If f possesses a second derivative f'', which is measurable and bounded on $(0,\infty)$, then

$$\lim_{n\to\infty} n\left(M_{mn}\left(f\left(\frac{u}{n}\right),nx\right) - G_m(f(u),x)\right) = \frac{1}{2m\Gamma(m)}\left(\frac{m}{x}\right)^m\int_0^{\infty} uf''(u)u^{m-1}e^{-mu/x}du. \tag{1.10}$$

Where $\omega_2(f,\delta)$ is the second modulus of continuity of f.

In this note, we will study the following four issues:

- Provide explicit semigroup representation formulae for above Bernstein-type operators.
- Improve the upper bound in (1.6), (1.7), and (1.8) partially.
- Deduce the so-called Voronovskaja-type result that Bernstein-type operators converge to szász operator ((1.6),(1.7),(1.8)) like (1.10).
- Generalize relevant results to some cases of integral operators, such as Kantorovich-type operators, Durrmeyer-type operators and Hybrid integral operators.

To deal with the aforementioned issues, we adopt a new technique that comes from the representation of operator semigroups. To our knowledge, it is the first time for this technique to be used in the research issue of approximating linear operator by other linear operators.

First, we are concerned about a sequence space (see [19]). For $l > 0$, $k = 0,1,2,\ldots$, $f_l(k) := f(\frac{k}{l})$, consider

$$\Omega := \left\{f_l = (f_l(0), f_l(1), f_l(2), \ldots)\,\middle|\, f \in C_B[0,+\infty), \|f_l\| = \sup_{k\in\mathbb{N}}|f_l(k)| < \infty\right\}.$$

A linear contraction B on Ω is defined by $(Bf_l)(k) = f_l(k+1)$, $k = 0,1,2,\ldots$, here B is correlate with l, we omit it for convenience without confusion, thereby $A = B - I$ is the generator of the Poisson convolution semigroup $T(t) =: e^{tA}$, $t \ge 0$, we can see that $T(t)$ is contractive semigroup. For the infinitesimal generator A, let $A^r (r \ge 1)$ denotes the rth power of A with domain $D(A^r)$.