Applied Mathematics and Computation 271 (2015) 991–1003

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On Chlodowsky variant of Szász operators by Brenke type polynomials

M. Mursaleen [a,b,*], Khursheed J. Ansari [a]

[a] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[b] Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

A R T I C L E I N F O

MSC:
41A10
41A25
41A36

Keywords:
Szász operators
Modulus of continuity
Rate of convergence
Weighted space
Divided differences
Brenke polynomials

A B S T R A C T

The aim of the present paper is to give a Chlodowsky type generalization of Szász operators defined by means of the Brenke type polynomials. We obtain convergence properties of our operators with the help of universal Korovkin-type property and also establish the order of convergence by using a classical approach, the second order modulus of continuity and Peetre's *K*-functional. We also give a Voronoskaja type theorem. Furthermore, we study the convergence of these operators in a weighted space of functions on a positive semi-axis. Some graphical examples for the convergence of our operators and error estimation are also given.

1. Introduction and preliminaries

Recently, Varma et al. [19] constructed positive linear operators with the help of Brenke type polynomials. Brenke type polynomials [5] have generating functions of the form

$$A(t)B(xt) = \sum_{k=0}^{\infty} p_k(x)t^k \tag{1.1}$$

where A and B are analytic functions:

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \tag{1.2}$$

$$B(t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0\,(r \geq 0) \tag{1.3}$$

and have the following explicit relation:

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \quad k = 0, 1, 2, \ldots. \tag{1.4}$$

* Corresponding author at: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India. Tel.: +0091 571 2720241.
E-mail addresses: mursaleenm@gmail.com, mursaleen.mm@amu.ac.in (M. Mursaleen), ansari.jkhursheed@gmail.com (K.J. Ansari).

http://dx.doi.org/10.1016/j.amc.2015.08.123
0096-3003/

Using the following restrictions:

(i) $A(1) \neq 0, \ \frac{a_{k-r}b_r}{A(1)} \geq 0, \ \ 0 \leq r \leq k, \ \ k = 0, 1, 2, \ldots,$
(ii) $B: [0, \infty) \to (0, \infty)$,
(iii) (1.1) and the power series (1.2) and (1.3) converge for $|t| < R \ (R > 1)$.

Varma et al [19] introduced the following positive linear operators involving the Brenke type polynomials

$$L_n(f; x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{1.5}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

Let $B(t) = e^t$ and $A(t) = 1$. We meet the following Szász [18] operators

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1.6}$$

In this paper, we consider the Chlodowsky [6] variant of Szász type operators involving the Brenke polynomials [17] given by (1.5) as follows:

$$L_n^*(f; x) := \frac{1}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right) \tag{1.7}$$

where (b_n) is a positive increasing sequence such that

$$\lim_{n\to\infty} b_n = \infty, \ \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{1.8}$$

and p_k are Brenke polynomials defined by (1.1). For another Chlodowsky type variation of positive linear operators, one can refer to [4].

The rest of the paper is organized as follows. In Section 2 we obtain some local approximation results by the generalized Szász operators given by (1.7). In particular, the convergence of operators is examined with the help of Korovkin's theorem. The order of approximation is established by means of a classical approach, the second-order modulus of continuity and Peetre's K -functional. Section 3 is devoted to study some convergence properties of these operators in weighted spaces with weighted norm on the interval $[0, \infty)$ by using the weighted Korovkin-type theorems [9,10]. Some examples are also given to compute error estimation by modulus of continuity in Section 4.

Note that throughout the paper we will assume that the operators L_n^* are positive and we use the following test functions

$$e_i(x) = x^i, \quad i \in \{0, 1, 2, 3, 4\}.$$

Also we consider

$$\lim_{y\to\infty} \frac{B^{(k)}(y)}{B(y)} = 1, \ \text{for } k \in \{1, 2, 3, \ldots, r\}. \tag{1.9}$$

We suggest the readers for further motivation by including citations of some recent papers on the subject dealing analogously with (for example) different classes of polynomials as well as Korovkin-type approximation theorems and Voronovskay-type approximation theorems (cf. [2,3,8,13–16]).

2. Local approximation properties of $L_n^*(f; x)$

We denote by $C_E[0, \infty)$ the set of all continuous functions f on $[0, \infty)$ with the property that $|f(x)| \leq \beta e^{\alpha x}$ for all $x \geq 0$ and some positive finite α and β. For a fixed $r \in \mathbb{N}$, we denote by $C_E^r[0, \infty) = \{f \in C_E[0, \infty) : f', f'', \ldots, f^{(r)} \in C_E[0, \infty)\}$. Using equality (1.1) and the fundamental properties of the L_n^* operators, one can easily get the following lemmas:

Lemma 2.1. *For all $x \in [0, \infty)$, we have*

$$L_n^*(e_0; x) = 1; \tag{2.1}$$

$$L_n^*(e_1; x) = \frac{B'\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)} x + \frac{b_n}{n}\frac{A'(1)}{A(1)}; \tag{2.2}$$

$$L_n^*(e_2; x) = \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)} x^2 + \frac{b_n}{n}\frac{\left(A(1) + 2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)} x + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)}; \tag{2.3}$$