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A solution of two-parameter asymptotic expansions for a two-dimensional unsteady boundary layer

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ABSTRACT

A solution procedure based on two-parameter asymptotic expansions, in terms of a Blasius parameter and a dimensionless time, is presented for a two-dimensional, unsteady boundary layer over a flat surface. The Blasius parameter is used to scale the stretching of the boundary-layer length scale, and the dimensionless time represents the unsteadiness caused by the outer flow field. The matching conditions between the outer solutions and inner solutions are obtained according to the matching procedure from which the streamfunction, velocity and pressure are matched all at the same time. Closed-form solutions are obtained until the second-order expansions of the solution. Applications of the solution to example problems are given with comparisons to the results in the literature to show the validity and versatility of the current solution to accommodate a variety of outer flows. The solution is even valid for predicting the time and location when the flow separation first occurs in some applications.

1. Introduction

Analytical solutions for unsteady laminar flow over a flat plate have been studied in the literature. Schlichting [1] listed several references for particular analytical solutions for unsteady boundary layer. For periodic external flow, the method by Lin [2] can effectively lead to the solutions. For external flow with perturbed unsteady motion, Lighthill [3] has proposed series expansions, which can be used in this study. The similarity or semi-similarity solution methods seek a boundary-layer solution with reduced independent variables (see [4–7]). However, the existence of similarity solutions depends on the form of external flow (i.e. outer flow in terms of asymptotic expansions), with transformations that satisfy the Lie group-invariant similarity [6,8].

In this study, similarity solutions for a two-dimensional unsteady boundary layer are sought in a general form of (with all the variables here and in the entire paper to be dimensionless)

$$u(x,y,t) = U(x,0,t)H(\eta,t) \tag{1.1}$$

where $\eta = y/N(x,t)$ is the inner length variable, $N(x,\,t)$ is the inner length scale function, and $U(x,\,0,\,t)$ is the external flow x-velocity on the boundary. The inner solution, $H(\eta,t)$, is sought using the two-parameter asymptotic expansions. Since the solution in Eq. (1.1) is constructed for any form of outer flow, a complete similarity solution is not pursued in the sense of satisfying the Lie group-similarity invariant property under a transformation. For this reason, instead of seeking a particular transformation, a simple, physically meaningful transformation is employed, by using the Blasius parameter, $\epsilon = \sqrt{t/Re}$ where Re is the Reynolds

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Number. While the Blasius parameter is used to scale the stretching of the boundary-layer length scale, a second parameter, a dimensionless time, accommodates the unsteadiness related to the unsteady outer flow (the potential flow in this study). The dimensionless time is characterized by the outer flow velocity and length scales. This dimensionless time is shared by both the outer and inner solutions, as in the boundary layer, the length scale changes but the time scale remains the same. In this way, the solution for Eq. (1.1) is obtained for each order of ϵ, and expanded for each order of dimensionless time. Similar two-parameter asymptotic expansions for boundary-layer solutions have been used in several applications (see [9–11]). Although the time series expansion restricts the solution to be valid only for short time duration, the examples presented in this study show that the time of flow separation occurrence can be predicted even when the time is not very small.

In the following discussion, only the procedure of solving the first two orders of solutions in both of the parameters is presented. This is because at orders up to, and higher than $O[\epsilon^2]$, the viscous effect needs to be included in the outer solution. Therefore, for the boundary-layer type solutions where the outer flow is considered inviscid, solutions with orders higher than $O[\epsilon^1]$ are not necessary. However, higher-order solutions in time can be achieved following the same procedure presented in this study, with the possibility that numerical solutions may have to be employed instead of the closed-form solution because of the increased complexity of the equations for the higher-order solutions.

The solution format of the asymptotic expansions starts with the streamfuction and velocity and is given in Section 2 for each order of ϵ. The governing equations for both the inner and outer flow are derived in Section 3, with the matching conditions obtained by switching outer and inner variables [12]. Since the pressure can be derived in incompressible flows when the velocity field is known, pressure matching is accomplished through the governing equations. Thus, the pressure distribution is developed in Section 4 after the governing equations for the inner and outer regions are discussed. Subsequently, closed form solutions for each term of the two-parameter expansions are developed in Sections 5 and 6. Finally, several application examples are discussed and compared with the literature results in Section 7, including boundary-layer formation after impulsive start or accelerated motion, and boundary-layer separation for flow induced by a single vortex or a pair of counter-rotation vortices over a flat plate. In addition, an earlier format of the lower-order solution has been used previously to provide a smooth initial condition for numerical simulation of vortex flow near a wall boundary [13].

It should be noted that the complexities of unsteady, two dimensional, boundary layer flow with a prescribed external velocity profile is illustrated in the literature, e.g., in [14–19]. The current study is not intended to tackle all these aspects of the problem. Rather, the analytical form of matched asymptotic expansions presented here can be used to shed some lights on these issues.

2. The solution format

The appropriate streamfunction for the outer flow can be written as

$$\Psi = \Psi_0(x, y, t) + \epsilon\Psi_1(x, y, t) + O[\epsilon^2] \tag{2.1}$$

with,

$$U = \frac{\partial\Psi}{\partial y} = \frac{\partial\Psi_0}{\partial y} + \epsilon\frac{\partial\Psi_1}{\partial y} + O[\epsilon^2] \tag{2.2}$$

and,

$$V = -\frac{\partial\Psi}{\partial x} = -\frac{\partial\Psi_0}{\partial x} - \epsilon\frac{\partial\Psi_1}{\partial x} + O[\epsilon^2] \tag{2.3}$$

where U and V are, respectively, x and y direction velocities of external flow. This streamfunction must satisfy the unsteady potential flow condition and boundary conditions.

Similarly, the inner flow is represented as

$$\psi = 2\epsilon[\psi_0(x, \eta, t) + \epsilon\psi_1(x, \eta, t)] + O[\epsilon^3] \tag{2.4}$$

where $\eta = \frac{y}{2\epsilon}$ is the inner stretched vertical coordinate. Following Blasius' solution [1], 2ϵ is used instead of ϵ for algebraic simplicity in the matching. Here, the inner velocity components are

$$u = \frac{\partial\psi}{\partial y} = 2\epsilon\left(\frac{1}{2\epsilon}\frac{\partial\psi_0}{\partial\eta} + \frac{1}{2}\frac{\partial\psi_1}{\partial\eta}\right) + O[\epsilon^2] = \left(\frac{\partial\psi_0}{\partial\eta} + \epsilon\frac{\partial\psi_1}{\partial\eta}\right) + O[\epsilon^2] \tag{2.5}$$

and,

$$v = -\frac{\partial\psi}{\partial x} = -2\epsilon\left(\frac{\partial\psi_0}{\partial x} + \epsilon\frac{\partial\psi_1}{\partial x}\right) + O[\epsilon^3] \tag{2.6}$$

This streamfunction must satisfy the governing equation for viscous flow and the no-slip boundary conditions.

Following the generalized matching principle [12], which requires that the inner and outer asymptotic expansions match at their overlapping limits, the outer expansion is written in terms of the inner variable η while the inner expansion is rewritten in the outer variable y. Subsequently, both the streamfunction and velocity expansions are rewritten and the corresponding terms matched. This matching procedure allows the streamfunction and velocity representations to be matched simultaneously along the overlap zone, and yields: