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A note on the computation of the extrema of Young's modulus for hexagonal materials: An approach by planar tensor invariants

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ABSTRACT

A simple method to obtain the highest and lowest Young's modulus for a material of the hexagonal class is presented. It is based upon the use of tensor invariants of plane anisotropic elasticity; in fact, the cylindrical symmetry of the elastic tensor allows for transforming the 3D original problem into a planar one, with a considerable simplification.

1. Introduction

The problem of finding the extrema of the Young's modulus for materials belonging to the hexagonal class has been recently addressed by some authors, [1–3]. In particular, the last work by Cazzani clearly states some historical contributions going back to the XIX century, namely to the works of Saint Venant, [4–6].

The purpose of this paper is to show that, thanks to the particular elastic symmetry of hexagonal materials, the problem of finding the extrema of the Young's modulus, as well as of any other elastic parameter, can be found by a direct bi-dimensional approach, much easier and straightforward than the general approaches proposed in the cited papers. Such an approach is based upon the *polar formalism*, where tensor invariants and angular variables are used to represent the elastic behavior of the material, in a way that reveals to be really powerful to state and solve the problem.

2. Statement of the problem by the polar formalism

Let us consider an elastic material having a hexagonal symmetry with X_3 as the axis of elastic symmetry. We address here the question of determining the extrema of the Young's modulus.

From a purely elastic point of view, a hexagonal symmetry cannot be distinguished from transverse isotropy: they share the same type of elastic tensors. This means also that for a hexagonal material, the elastic behavior is the same in all the meridian planes, i.e. those containing the X_3 axis.

For the purposes of this paper, this implies that it is sufficient to study the Young's modulus variation in any of such planes: the problem can be completely reduced from a three-dimensional to a planar one. For the sake of simplicity, we will indicate by

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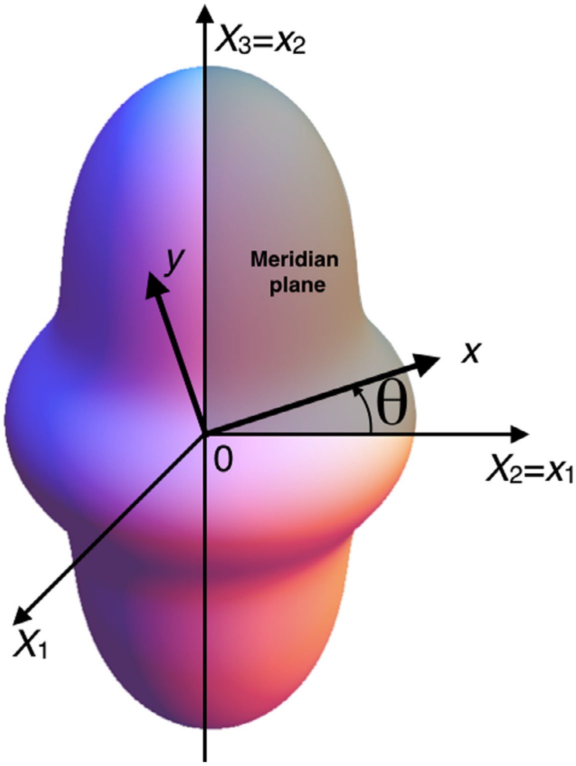

Fig. 1. Sketch of the reference frames.

x and y a couple of Cartesian axes in such a plane. Unlike what is commonly done in the literature, where the axis of symmetry for a material of the hexagonal symmetry class is labelled as the axis of X_3, the remaining two axes of X_1 and X_2 laying in the transversal plane, the use of axes labelled x and y has been preferred here, for emphasizing the planar character of the approach. These axes lay in a meridian plane; the axis of x is rotated through an angle θ with respect to the transversal plane, hence, for instance, with respect to the X_1 or X_2 axes. For the sake of shortness, and for giving a particular importance to the couple of axes of orthotropy, we will name x_1 and x_2 the axes of x and y, respectively, when $\theta = 0$. Hence, for being more precise, $x_1 = X_1$ (or, indifferently, X_2) and $x_2 = X_3$, see Fig. 1. To remark that angle θ is here the *latitude*, i.e. $\theta = 0$ corresponds to the equator and $\theta = \pi/2$ to the pole.

It is worth to recall that the components of the compliance tensor $\mathbb{S}$ are exactly the same as those of the corresponding material in a planar state of stress. As $E(\theta)$ is given by (Voigt's notation is used throughout all this paper)

$$E(\theta) = \frac{1}{S_{xx}(\theta)}, \tag{1}$$

the final question is hence reduced to a planar elastic one, depending on only one variable: to find the extrema of the Young's modulus $E(\theta)$ of a planar orthotropic material. For the mathematical symmetries, the study can be restricted to the set $0 \le \theta < \pi/2$.

An interesting tool for the study of planar problems in anisotropic elasticity is the polar formalism, introduced by Verchery in 1979, [7]. This method is particularly effective in several situations, namely because a tensor is described making use of invariants and angles only, which reveals to be very useful for describing anisotropic properties. Restricting here the attention to the strict topic of this paper (the interested reader can find a detailed account of the polar formalism in [8]; here, due to the peculiarity of the matter, only the case of orthotropic materials is considered), it is

$$S_{xx}(\theta) = t_0 + 2t_1 + (-1)^k r_0 \cos 4(\varphi_1 - \theta) + 4r_1 \cos 2(\varphi_1 - \theta), \tag{2}$$

with t_0 and t_1 the *isotropy invariants*, r_0, r_1 and k the *anisotropy invariants*; all of them are non-negative quantities. The last polar parameter, φ_1, is an angle that fixes the frame.

The expressions of the above quantities are given as functions of the Cartesian components of $\mathbb{S}$ by the following relations (with the usual frame $\{X_1, X_2, X_3\}$ mentioned above, S_{22} should be S_{33}, S_{12} should be S_{13} or S_{23} and S_{66} indifferently S_{44} or S_{55}, while S_{11} does not change of indexes):

$$t_0 = \frac{1}{8}(S_{11} - 2S_{12} + S_{66} + S_{22}),$$