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Closed formulas for computing higher-order derivatives of functions involving exponential functions

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ABSTRACT

For integers $k \geq 1$ and $n \geq 0$, the functions $1/(1-\lambda e^{\alpha t})^k$ and the derivatives $(1/(1-\lambda e^{\alpha t}))^{(n)}$ can be expressed each other by linear combinations. Based on this viewpoint, we find several new closed formulas for higher-order derivatives of trigonometric and hyperbolic functions, derive a higher-order convolution formula for the tangent numbers, and generalize a recurrence relation for the tangent numbers.

1. Introduction

Motivated by two identities in [10] the following problem was posed in the recently published paper [6]. For $t \neq 0$ and $k \in \mathbb{N}$, determine the numbers $a_{k,i-1}$ for $1 \leq i \leq k$ such that

$$\frac{1}{(1-e^{-t})^k} = 1 + \sum_{i=1}^{k} a_{k,i-1}\left(\frac{1}{e^t-1}\right)^{(i-1)}. \tag{1.1}$$

Stimulated by this problem, eight identities involving an exponential function were established in [6]. The authors [15] answered this question alternatively with combinatorial technique and unified the eight identities due to Guo and Qi to two identities involving two parameters. We state them as the following theorems:

Theorem 1.1. *Let $n \geq 0$ be an integer and let α and λ be two real parameters. Then*

$$\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(n)} = \sum_{k=1}^{n+1} \frac{\alpha^n(-1)^{n-k+1}(k-1)!S(n+1,k)}{(1-\lambda e^{\alpha t})^k}, \tag{1.2}$$

where $S(n,k)$ are the Stirling numbers of the second kind.

Theorem 1.2. *Let $n \geq 1$ be an integer and let α and λ be two real parameters. Then*

$$\frac{1}{(1-\lambda e^{\alpha t})^n} = \sum_{k=1}^{n} \frac{(-1)^{n-k}s(n,k)}{(n-1)!\alpha^{k-1}}\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(k-1)}, \tag{1.3}$$

where $s(n,k)$ are the Stirling numbers of the first kind.

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The above two theorems imply that the functions $1/(1-\lambda e^{\alpha t})^k$ and the derivatives $(1/(1-\lambda e^{\alpha t}))^{(n)}$ can be expressed each other by linear combinations. Guo and Qi [7] gave elementary proofs for these two theorems.

Finding closed expressions for higher-order derivatives of trigonometric functions is a subject of recurrent interest. In a recent paper, Adamchik [1] solved a long-standing problem on finding a closed-form expression for the higher-order derivatives of the cotangent function by showing that

$$\frac{d^n}{dx^n}\cot x=(2i)^n(\cot x-i)\sum_{k=1}^{n}\frac{k!}{2^k}S(n,k)(i\cot x-1)^k, \tag{1.4}$$

where $i=\sqrt{-1}$. Independently, Boyadahiev [2] derived derivative polynomials for the hyperbolic and trigonometric tangent, cotangent and secant in explicit form.

Using basic operations on the Zeon algebra, a simple and short proof of Formula (1.4) was given in [11]. Recently, by using the derivative polynomials introduced by Hoffman [9], Cvijović [8] derived closed-form higher derivative formulas for eight trigonometric and hyperbolic functions, which involve the Carlitz-Scoville higher-order tangent and secant numbers [3,4]. More recently, Qi [12] found explicit formulas for higher order derivatives of the tangent and cotangent functions as well as powers of the sine and cosine functions, obtained explicit formulas for two Bell polynomials of the second kind for successive derivatives of sine and cosine functions, presented curious identities for the sine function and discovered explicit formulas and recurrence relations for the tangent numbers, the Bernoulli numbers, the Genocchi numbers, special values of the Euler polynomials at zero, and special values of the Riemann zeta function at even numbers. Moreover, in [12] some comments on five different forms of higher order derivatives for the tangent function and on derivative polynomials of the tangent, cotangent, secant, cosecant, hyperbolic tangent, and hyperbolic cotangent functions were given.

This paper is a sequel to the work of Xu and Cen [15]. By directly applying Theorems 1.1 and 1.2, we will find several new explicit expressions for higher-order derivatives of trigonometric and hyperbolic functions which are different from those of Adamchik and Boyadahiev. By means of a key equality in [15], we present a proof to show that they are equivalent. Furthermore, we derive a higher-order convolution formula for the tangent numbers, which generalizes the recurrence relation for the tangent numbers given in [12].

2. Main results

First of all, recall briefly that Stirling numbers of two kinds play very important roles in combinatorial analysis and number theory. Let $S(n, k)$ be the Stirling number of the second kind and let $s(n, k)$ be the Stirling number of the first kind. The Stirling number of the second kind $S(n, k)$ counts the number of partitions of a set of n elements into k indistinguishable boxes in which no box is empty, and the Stirling number of the first kind $s(n, k)$ counts the number of arrangements of n objects into k nonempty circular permutations. The Stirling numbers of two kinds satisfy a Pascal-like recurrence relation

$$S(n,k)=kS(n-1,k)+S(n-1,k-1), \tag{2.1}$$

$$s(n,k)=-(n-1)s(n-1,k)+s(n-1,k-1). \tag{2.2}$$

See [13,14]. The exponential generating function of $S(n, k)$ is the formal power series

$$\frac{(e^t-1)^k}{k!}=\sum_{n\ge k}S(n,k)\frac{t^n}{n!}, \tag{2.3}$$

and the exponential generating function of $s(n, k)$ is the formal power series

$$\frac{1}{k!}(\ln(1+t))^k=\sum_{n\ge k}s(n,k)\frac{t^n}{n!}. \tag{2.4}$$

For more details on Stirling numbers of both kinds, see [5].

Now, by applying Theorem 1.1 we turn to present our formulas for the nth derivatives of trigonometric functions in terms of the Stirling numbers.

Theorem 2.1. *Let $n \ge 1$ be an integer. Then*

$$(\tan x)^{(n)}=(-i)^{n+1}\sum_{k=1}^{n+1}2^{n+1-k}(k-1)!S(n+1,k)(i\tan x-1)^k, \tag{2.5}$$

$$(\cot x)^{(n)}=i^{n+1}\sum_{k=1}^{n+1}(-1)^{n-k}2^{n+1-k}(k-1)!S(n+1,k)(i\cot x+1)^k. \tag{2.6}$$

Proof. Letting $\lambda=-1$ and $\alpha=2i$ in Theorem 1.1 yields

$$\left(\frac{1}{e^{2ix}+1}\right)^{(n)}=(2i)^n\sum_{k=1}^{n+1}\frac{(-1)^{n-k+1}(k-1)!S(n+1,k)}{(e^{2ix}+1)^k}. \tag{2.7}$$