Applied Mathematics and Computation 270 (2015) 387–400

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A class of two-point sixth-order multiple-zero finders of modified double-Newton type and their dynamics

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ARTICLE INFO

MSC:
65H05
65H99
41A25
65B99

Keywords:
Multiple-zero finder
Extraneous fixed point
Double-Newton
Basins of attraction

ABSTRACT

Under the assumption of the known multiplicity of zeros of nonlinear equations, a class of two-point sextic-order multiple-zero finders and their dynamics are investigated in this paper by means of extensive analysis of modified double-Newton type of methods. With the introduction of a bivariate weight function dependent on function-to-function and derivative-to-derivative ratios, higher-order convergence is obtained. Additional investigation is carried out for extraneous fixed points of the iterative maps associated with the proposed methods along with a comparison with typically selected cases. Through a variety of test equations, numerical experiments strongly support the theory developed in this paper. In addition, relevant dynamics of the proposed methods is successfully explored for various polynomials with a number of illustrative basins of attraction.

1. Introduction

Root-finding of nonlinear equations of the form $f(x) = 0$ has been one of the most frequently occurring problems in scientific work. In rare cases, it is possible to solve the governing equations exactly. In most cases, however, only approximate solutions may resolve the real problems handling such as weather forecast, accurate positioning of satellite systems in the desired orbit, measurement of earthquake magnitudes and other high-level engineering technologies. Among simple-zero finders, the most widely accepted classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

solves $f(x) = 0$ without difficulty, provided that a good initial guess x_0 is chosen near the zero α. Under the assumption that the multiplicity m is known a priori, it is of considerable interest to design efficient methods for locating repeated zeros of $f(x)$. For the zero α with a given multiplicity of $m \geq 1$, modified Newton's method [36,37] in the following form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

is frequently used by many researchers. It is known that numerical scheme (1.2) is a second-order one-point optimal [25] method on the basis of Kung–Traub's conjecture [25] that any multipoint method [35] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. Other higher-order multiple-zero finders can be found in papers [15,17–20,26,27,40,45].

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http://dx.doi.org/10.1016/j.amc.2015.08.039

Given a known multiplicity of $m > 1$, we propose in this paper a family of new two-point sixth-order multiple-zero finders of modified double-Newton type by adding the second step to (1.2) of the form:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - Q_f(x_n)\cdot\dfrac{f(y_n)}{f'(y_n)}, \end{cases} \tag{1.3}$$

where the desired form of the weight function Q_f using only two-point functional information at x_n and y_n will be extensively studied for maximal order of convergence in Section 2.

This paper is divided into six sections. Investigated in Section 2 is methodology and convergence analysis for newly proposed multiple-zero finders. A main theorem is established to state convergence order of six as well as to derive asymptotic error constants and error equations by use of a family of bivariate weight functions Q_f dependent on two principal roots of function-to-function and derivative-to-derivative ratios. In Section 3, special forms of weight functions are considered based on polynomials and rational functions with labeled case numbers. Section 4 discusses the extraneous fixed points and related dynamics behind the basins of attraction. Tabulated in Section 5 are computational results for a variety of numerical examples. Table 6 compares the magnitudes of $e_n = x_n - \alpha$ among those of typically selected cases of the proposed methods. Dynamical properties of the proposed methods along with their illustrative basins of attraction are displayed with detailed analyses and comments. Overall conclusion as well as possible future work is briefly discussed at the end of the final section.

2. Methodology and convergence analysis

Let a function $f:\mathbb{C}\to\mathbb{C}$ have a repeated zero α with integer multiplicity $m > 1$ and be analytic [1] in a small neighborhood of α. Then, given an initial guess x_0 sufficiently close to α, new iterative methods proposed in (1.3) to find an approximate zero α of multiplicity m will take the specific form of:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - Q_f(u,s)\cdot\dfrac{f(y_n)}{f'(y_n)}, \end{cases} \tag{2.1}$$

where

$$u = \left[\frac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{2.2}$$

$$s = \left[\frac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}, \tag{2.3}$$

and where $Q_f:\mathbb{C}^2\to\mathbb{C}$ is holomorphic [21,39] in a neighborhood of $(0, 0)$. Since u and s are respectively a one-to-m and a one-to-(m-1) multiple-valued functions, we consider their principal analytic branches [1]. Hence, it is convenient to treat u as a principal root given by $u = \exp[\frac{1}{m}\mathrm{Log}(\frac{f(y_n)}{f(x_n)})]$, with $\mathrm{Log}(\frac{f(y_n)}{f(x_n)}) = \mathrm{Log}\left|\frac{f(y_n)}{f(x_n)}\right| + i\,\mathrm{Arg}(\frac{f(y_n)}{f(x_n)})$ for $-\pi < \mathrm{Arg}(\frac{f(y_n)}{f(x_n)}) \le \pi$; this convention of $\mathrm{Arg}(z)$ for $z\in\mathbb{C}$ agrees with that of $\mathrm{Log}[z]$ command of Mathematica [44] to be employed later in numerical experiments of Section 5. By means of further inspection of u, we find that $u = \left|\frac{f(y_n)}{f(x_n)}\right|^{\frac{1}{m}}\cdot\exp[\frac{i}{m}\,\mathrm{Arg}(\frac{f(y_n)}{f(x_n)})] = O(e_n)$. Similarly we treat $s = \left|\frac{f'(y_n)}{f'(x_n)}\right|^{\frac{1}{m-1}}\cdot\exp[\frac{i}{m-1}\,\mathrm{Arg}(\frac{f'(y_n)}{f'(x_n)})] = O(e_n)$. In addition, we find that $O(\frac{f(y_n)}{f'(y_n)}) = O(e_n^2)$.

Definition 2.1 (Error equation, asymptotic error constant, order of convergence)**.** Let $\{x_0, x_1, \ldots, x_n, \ldots\}$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p\in\mathbb{R}$ and $b\in\mathbb{R}-\{0\}$ such that the following error equation holds

$$e_{n+1} = b{e_n}^p + O(e_n^{p+1}), \tag{2.4}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [42].

In this paper, we investigate the maximal convergence order of proposed methods (2.1). We here establish a main theorem describing the convergence analysis regarding proposed methods (2.1) and find out how to construct the weight function Q_f for sextic-order convergence. Hence, it suffices to consider the weight function Q_f with $O(Q_f(u,s)) = O(e_n^4)$ due to the fact that $O(\frac{f(y_n)}{f'(y_n)}) = O(e_n^2)$.

Applying the Taylor's series expansion of f about α, we get the following relations:

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!}{e_n}^m[1 + \theta_2 e_n + \theta_3 e_n^2 + \theta_4 e_n^3 + \theta_5 e_n^4 + \theta_6 e_n^5 + \theta_7 e_n^6 + O(e_n^7)], \tag{2.5}$$