Applied Mathematics and Computation 270 (2015) 441–457

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Optimal global approximation of SDEs with time-irregular coefficients in asymptotic setting☆

Paweł Przybyłowicz*

*Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, Krakow 30-059, Poland*

ARTICLE INFO

*MSC:*
68Q25
65C30

*Keywords:*
Non-standard assumptions
Global approximation
Adaptive/nonadaptive standard information
Minimal error algorithm
Monte Carlo algorithms

ABSTRACT

We investigate strong approximation of solutions of scalar stochastic differential equations (SDEs) with irregular coefficients. In Przybyłowicz (2015) [23], an approximation of solutions of SDEs at a single point is considered (such kind of approximation is also called a *one-point approximation*). Comparing to that article, we are interested here in a global reconstruction of trajectories of the solutions of SDEs in a whole interval of existence. We assume that a drift coefficient $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous with respect to a space variable, but only measurable with respect to a time variable. A diffusion coefficient $b : [0, T] \to \mathbb{R}$ is only piecewise Hölder continuous with Hölder exponent $\varrho \in (0, 1]$. The algorithm and results concerning lower bounds from Przybyłowicz (2015) [23] cannot be applied for this problem, and therefore we develop a suitable new technique. In order to approximate solutions of SDEs under such assumptions we define a discrete type randomized Euler scheme. We provide the error analysis of the algorithm, showing that its error is $O(n^{-\min\{\varrho,1/2\}})$. Moreover, we prove that, roughly speaking, the error of an arbitrary algorithm (for fixed $a$ and $b$) that uses $n$ values of the diffusion coefficient, cannot converge to zero faster than $n^{-\min\{\varrho,1/2\}}$ as $n \to +\infty$. Hence, the proposed version of the randomized Euler scheme achieves the established best rate of convergence.



## 1. Introduction

We consider global approximation of the following scalar stochastic differential equations

$$\begin{cases} dX(t) = a(t, X(t))dt + b(t)dW(t), & t \in [0, T], \\ X(0) = \eta, \end{cases} \tag{1}$$

where $T > 0$, $W = \{W(t)\}_{t\in[0,T]}$ is a standard one-dimensional Brownian motion on some probability space $(\Omega, \Sigma, \mathbb{P})$ and an initial-value $\eta$ is independent of $W$. We assume that the drift coefficient $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in $\mathbb{R}$ with respect to space variable. On the other hand, we assume that $a$ is only *measurable* with respect to time variable, while $b$ is only *piecewise Hölder continuous* in $[0, T]$ with Hölder exponent $\varrho \in (0, 1]$. Such assumptions are sufficient in order to assure that (1) has a unique strong solution $X = \{X(t)\}_{t\in[0,T]}$, see for example Section 2.2 and Theorem 3.1 in [16], Theorem 4.5.3 in Section 4.5 in [14], Section 5.2 and Theorem 2.9 in [13] or discussion at pages 11–13 in [17] where also further references are provided. The

☆ This research was partially supported by the Polish NCN grant—decision no. DEC-2013/09/B/ST1/04275 and by AGH local grant.

* Tel.: +48 12 617 44 05.
*E-mail address:* pprzybyl@agh.edu.pl

http://dx.doi.org/10.1016/j.amc.2015.08.055
0096-3003/

efficient approximation of $X$ with the (asymptotic) error as small as possible is of interest, since the analytic solutions are known only in particular cases.

In the case of ordinary differential equations (ODEs i.e., problem (1) with $b \equiv 0$) for instance, Carathéodory ordinary differential equations (CODEs) were considered in [7,24,25]. In that papers suitable Monte Carlo methods were defined to approximate the solutions. In [11,12] the authors constructed optimal deterministic algorithms for initial-value problems with right-hand side functions $a$ that have discontinuous partial derivatives.

There is a rich literature that deals with approximation of stochastic differential equations (SDEs) with regular coefficients $a$ and $b$, see for example [14], which is the standard reference, and [17]. On the other hand, much less is known in case of SDEs with irregular coefficients. We recall some recently obtained results on the topic. In [27] weak convergence of the Euler algorithm was shown for SDEs with discontinuous $a$ and $b$. However, order of convergence has not been investigated. In [15] the authors also established rate of weak convergence of the Euler scheme for irregular drift coefficient $a = a(t, y)$ and sufficiently smooth diffusion $b$. Strong convergence of the Euler scheme applied to SDEs with discontinuous drift coefficients has been studied in [18], while strong convergence of the drift-implicit square-root Euler approximations has been established in [4]. Nevertheless, the optimality of the proposed algorithms has not been investigated in the mentioned articles. Optimal rates of convergence of the Euler scheme for SDEs (1), with coefficients $a = a(t, y)$ and $b = b(t)$ having separated variables and finite number of discontinuities with respect to the time variable $t$, have been shown in [21]. In [22] the authors investigated the error and worst-case optimality of the randomized Euler scheme applied to (1) with non-smooth $a$ and $b$. Finally, in [23] the author investigated asymptotic errors for the so called *strong one-point approximation* of SDEs (1) with time-irregular coefficients $a$ and $b$. In the one-point approximation we look for a random variable $\hat{X}(T)$ that is close (in the $L^q(\Omega)$-norm, $q \in [1, +\infty)$) to $X(T)$, see the discussion at page 2 in [17].

In [21,22] a one-point approximation of (1) was investigated. The error of an algorithm was meant there as the largest value of the averaged difference between the actual and computed approximation over a certain class of input data $(a, b, \eta)$. Such error setting is called the *worst-case setting*, see Chapter 4 in [26]. On the other hand, in [23] the one-point approximation of (1) was investigated in the *asymptotic setting*. This means that the error was considered to be the averaged difference between the real value and computed solution for a *fixed* $(a, b, \eta)$ from a certain set of input data. Then the (asymptotic) behavior of the error was investigated as a number of evaluations of $a$, $b$ and $W$ tended to infinity. (Such approach was mostly used in textbooks and articles on approximating SDEs.) The further problem was to establish lower bounds on the error of an arbitrary algorithm. We meant by that establishing the existence of a set of 'difficult' input $(a, b, \eta)$ for which the rate of convergence cannot be improved. We also wanted to know how 'large' (in a certain mathematical sense) a subset of such mappings $(a, b, \eta)$ was, see [8] and Chapter 10 in [26] for a further discussion. An algorithm with the best convergence properties was referred to as the *minimal error algorithm*. (For SDEs with smooth drift and diffusion coefficients the minimal asymptotic errors have been established in [5,6,17].)

In this paper we analyze *strong global approximation* of (1) in the *asymptotic setting*. We aim at the reconstruction of the whole trajectories of $X$ with an arbitrary sequence of approximations $\hat{X} = \{\hat{X}_n\}_{n\in\mathbb{N}}$, where each $\hat{X}_n$ uses $n$ samples of (piecewise Hölder continuous) diffusion coefficient $b$. The $n$th error is meant as the distance between $X$ and $\hat{X}_n$ measured in $L^q([0, T] \times \Omega)$-norm, $q \in [1, +\infty)$. Hence, it is an averaged distance, over all trajectories, between trajectories of $X$ and $\hat{X}_n$. We investigate behavior of the asymptotic error as $n \to +\infty$. (We give a detailed description of our goal in the next section.)

We now point out main difficulties which we have to handle in the paper. In the used setting an application of the randomized Euler algorithm $\hat{X}_n^{RE}$ from [22] is not possible, since it gives the approximation of the solution $X = X(t)$ only at finite number of discrete points $t_1, t_2, \dots, t_n \in [0, T]$. Also the continuous version $\tilde{X}_n^{RE}$ of $\hat{X}_n^{RE}$ (see (46) in [22]) is not implementable, since it requires the values of $W = W(t)$ for all $t \in [0, T]$. Due to low regularity of $a$ and $b$, we cannot use the algorithms developed in [5,6,14,17]. Moreover, it turns out that the lower bounds obtained in [21,22] (worst-case setting) and in [23] (asymptotic setting) for the one-point approximation do not imply in any way lower bounds for a global approximation of SDEs with the error measured in the $L^q([0, T] \times \Omega)$-norm. Finally, we cannot directly use the lower bounds on the error developed in [5,6,17] for the global approximation of SDEs to the whole class of $(a, b, \eta)$ considered in this paper, since we assume that the coefficients $a = a(t, y)$ and $b = b(t)$ may be non-smooth with respect to the time variable $t$ and only Lipschitz continuous with respect to the space $y$.

In order to reconstruct the solution $X = X(t)$ of (1) for all $t \in [0, T]$, we define a discrete type randomized Euler algorithm, denoted by $X^{RE} = \{X_n^{RE}\}_{n\in\mathbb{N}}$. We analyze its error and cost in terms of number of evaluations of $a$, $b$ and $W$. (See [2,3,7,10,24,25] where suitable versions of $X^{RE}$ were used for approximation of ODEs.) In order to deliver the corresponding asymptotic lower bounds on the error of an arbitrary algorithm, we extend the approach used in [23] for one-point approximation. As in the article [23] we use some general results obtained in [8,9] for the asymptotic setting. Moreover, we give a proof of a generalization of Theorem 3.3 (i) from [23], that concerns with an approximation of nonlinear operators defined on a nonempty subset of $\mathcal{M}_\infty([0, T])$ with values in a normed linear space (see Theorem 4.3). Such generalization for global approximation turns out to be necessary (see Remark 4.3).

The main result states that the error of an arbitrary sequence of approximations $\{\hat{X}_n\}_{n\in\mathbb{N}}$, where each $\hat{X}_n$ uses $n$ nonadaptive evaluations of a diffusion coefficient, cannot go to zero faster than $n^{-\min\{\varrho, 1/2\}}$ as $n \to +\infty$ (Theorem 4.5). This holds except on the *small* subset of the set of $b$'s under consideration. By the *small* subset we mean "the set of empty interior" (Theorem 4.2) or "of Lebesgue measure zero" (Theorem 4.4). Moreover, the discrete type randomized Euler algorithm $X^{RE}$ turns out be a method with optimal convergence properties (Theorem 4.5).

The structure of the paper is as follows. Problem formulation, assumptions and basic definitions are given in Section 2. Section 3 contains definition and detailed error analysis of the discrete type randomized Euler scheme $X^{RE}$ that is used in or-