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# Weighted pre-orders involving the generalized Drazin inverse ☆

Dijana Mosić*, Dragan S. Djordjević

*Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*

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ABSTRACT

The aim of this paper is to characterize new pre-orders defined on the set of all bounded linear operators between two Banach spaces. Thus, recent results on pre-orders involving the Drazin inverse of a complex matrix are extended to a more general setting.



## 1. Introduction

Let $\mathcal{A}$ be a Banach algebra. An element $b \in \mathcal{A}$ is an inner inverse of $a \in \mathcal{A}$ if $aba = a$. By $a\{1,5\}$ we denote the set of all inner inverse of $a$ which commutes with $a$, i.e. $a\{1,5\} = \{b \in \mathcal{A} : aba = a,\ ab = ba\}$.

Then $a \in \mathcal{A}$ is quasipolar if and only if there exists $b \in \mathcal{A}$ such that

$$ab = ba, \quad bab = b \quad \text{and} \quad a - aba \text{ is quasinilpotent.}$$

The element $b$, in the case when it exists, is unique and it is called the generalized Drazin inverse, or the Koliha-Drazin inverse of $a$, denoted by $a^d$ [7, Theorem 7.5.3], [12]. By $\mathcal{A}^d$ we denote the set of all generalized Drazin invertible elements of $\mathcal{A}$.

If the element $a - aba$ is nilpotent in the above definition, then $a^d = a^D$ is the ordinary Drazin inverse. The condition $a - a^2b$ is nilpotent is equivalent to $a^{k+1}b = a^k$, for some non-negative integer $k$. The smallest $k$ such that $a^{k+1}b = a^k$ holds, is called the index of $a$ and it is denoted by $ind(a)$. If $ind(a) \leq 1$, then $a$ is group invertible and $a^D$ is the group inverse of $a$ denoted by $a^\#$. The basic theory of the Drazin inverse and various applications can be found in the books [3,4].

Let $X$ and $Y$ denote arbitrary Banach spaces, and let $\mathcal{B}(X,Y)$ denote the set of all bounded linear operators from $X$ to $Y$. Set $\mathcal{B}(X) = \mathcal{B}(X,X)$. For an operator $A \in \mathcal{B}(X,Y)$, the symbols $N(A)$ and $R(A)$, respectively, will denote the null space and the range of $A$. A projection is a bounded linear operator $P \in \mathcal{B}(X)$ such that $P^2 = P$.

If $A \in \mathcal{B}(X)$ is a generalized Drazin invertible operator, then the spectral idempotent $A^\pi$ of $A$ corresponding to $\{0\}$ is given by $A^\pi = I - AA^d$. The matrix forms of operators $A$ and $A^d$ with respect to the space decomposition $X = N(A^\pi) \oplus R(A^\pi)$ are given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad A^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

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* Corresponding author. Tel.: +38118533014.
*E-mail addresses:* dijana@pmf.ni.ac.rs (D. Mosić), dragan@pmf.ni.ac.rs (D.S. Djordjević).

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where $A_1$ is invertible and $A_2$ is quasinilpotent [12, Theorem 7.1]. If we denote $C_A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q_A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}$, then $A = C_A + Q_A$ is known as the core-quasinilpotent decomposition of $A$. The operator $C_A$ is called the core part of $A$ and $Q_A$ is called the quasinilpotent part of $A$. Notice that $C_A = A^2A^d$ is group invertible, $C_A^\# = A^d$, $Q_A = AA^\pi$ and $C_AQ_A = 0 = Q_AC_A$ [12, Theorem 6.4].

Let $W \in \mathcal{B}(Y, X)$, and let $\mathcal{B}_W(X, Y)$ be the space $\mathcal{B}(X, Y)$ equipped with the multiplication $A * B = AWB$ and the norm $\|A\|_W = \|A\|\|W\|$. Then $\mathcal{B}_W(X, Y)$ becomes a Banach algebra [6]. $\mathcal{B}_W(X, Y)$ has the unit if and only if $W$ is invertible, and in this case $W^{-1}$ is the unit.

Let $W \in \mathcal{B}(Y, X)$ be a fixed nonzero operator. An operator $A \in \mathcal{B}(X, Y)$ is called Wg–Drazin invertible if $A$ is quasipolar in the Banach algebra $\mathcal{B}_W(X, Y)$. The Wg–Drazin inverse $A^{d,W}$ of $A$ is defined as the g–Drazin inverse of $A$ in the Banach algebra $\mathcal{B}_W(X, Y)$ [6].

Let us recall that if $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$ then the following conditions are equivalent [6]:

(1) $A$ is *Wg*-Drazin invertible,
(2) $AW$ is quasipolar in $\mathcal{B}(Y)$ with $(AW)^d = A^{d,W}W$,
(3) $WA$ is quasipolar in $\mathcal{B}(X)$ with $(WA)^d = WA^{d,W}$.

Then, the *Wg*-Drazin inverse $A^{d,W}$ of $A$ satisfies

$$A^{d,W} = ((AW)^d)^2A = A((WA)^d)^2.$$

The support idempotent $A^{\sigma,W}$ of $A$ is given by $A^{\sigma,W} = (AW)^dA = A(WA)^d$.

**Lemma 1.1** ([6]). *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)\backslash\{0\}$. Then $A$ is $W$ g-Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

*where $A_i \in \mathcal{B}(X_i, Y_i)$, $W_i \in \mathcal{B}(Y_i, X_i)$, for $i = \overline{1, 2}$, with $A_1$, $W_1$ invertible, and $W_2A_2$ and $A_2W_2$ quasinilpotent in $\mathcal{B}(X_2)$ and $\mathcal{B}(Y_2)$, respectively. The $W$ g-Drazin inverse of $A$ is given by*

$$A^{d,W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

*with $(W_1A_1W_1)^{-1} \in \mathcal{B}(X_1, Y_1)$ and $0 \in \mathcal{B}(X_2, Y_2)$.*

A binary relation on a non-empty set is called pre-order, if it is reflexive and transitive, and it is called a partial order relation if it is reflexive, antisymmetric and transitive. For some results of pre-orders and partial orders on the set of complex matrices see [14] and some related results can be found in the following references [8,10,11,13]. Some applications of pre-orders and partial orders you can find in [1,2].

Now, we will introduce the sharp order and the generalized Drazin pre-order on the corresponding subsets of $\mathcal{B}(X)$, which are extensions of similar orders for complex matrices [14].

Let $A, B \in \mathcal{B}(X)$ such that $ind(A) \le 1$. We define the sharp order in the following way

$$A \le^\# B \Leftrightarrow A^\#A = A^\#B \quad \text{and} \quad AA^\# = BA^\#.$$

**Lemma 1.2.** *[15, Theorem 3.3] The sharp order is a partial order on the set of operators $\{A \in \mathcal{B}(X) : ind(A) \le 1\}$.*

Let $A, B \in \mathcal{B}(X)$ be generalized Drazin invertible operators such that $A = C_A + Q_A$ and $B = C_B + Q_B$ are the core-quasinilpotent decompositions of $A$ and $B$ respectively. The operator $A$ is related to $B$ under the generalized Drazin relation (denoted by $A \le {}^dB$) if $C_A \le^\# C_B$.

**Theorem 1.1.** *Let $A, B \in \mathcal{B}(X)$ be generalized Drazin invertible operators such that $A = C_A + Q_A$ and $B = C_B + Q_B$ are the core-quasinilpotent decompositions of $A$ and $B$ respectively. Then $A \le {}^dB$ if and only if*

$$A^dA = A^dB \quad \text{and} \quad AA^d = BA^d.$$

**Proof.** If $A \le {}^dB$, then $C_AC_A^\# = C_BC_A^\# = C_A^\#C_B$. Now, we have

$$Q_BC_A^\# = Q_BC_A^\#C_AC_A^\# = Q_BC_BC_A^\#C_A^\# = 0$$

and similarly $C_A^\#Q_B = 0$. Thus,

$$AA^d = C_AC_A^\# = C_BC_A^\# = BC_A^\# = BA^d$$

and analogously $AA^d = A^dB$.

Assume that $A^dA = A^dB = BA^d$. With respect to the space decomposition $X = N(A^\pi) \oplus R(A^\pi)$, we can write $A$ and $A^d$ as in (1). Then

$$B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix} \quad \text{and} \quad B^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & B_2^d \end{bmatrix}.$$